EECS 205003 Session 3

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Overview of key ideas

vectors/matrices/subspaces

Vectors

Linear combination of vectors

$$x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{b}$$

Ex

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

collective of all multiples of u forms a line via origin

collective of all linear combination of $\mathrm{u}\ \&\ \mathrm{v}$ forms a plane

all linear combination of $\mathrm{u}, \mathrm{v}, \mathrm{w}$ forms a subspace

Matrices

coeff. matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

the product

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$ **(A is a difference matrix)**

($x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathrm{w}$: multiplying numbers by vectors

diff. view $\Updownarrow$

$A\mathrm{x}$: matrix multiplying numbers x_1, x_2, x_3 or A acts on vector x, the result is b: a combination of columns of A)

For any input x, the output of "multiplication" by A is some vector b

Ex:

$$A\begin{bmatrix}1\\4\\9\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}$$

A deeper question : For what x , does $A\mathrm{x}=\mathrm{b}$?

$$A\mathbf{x}=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_2-x_1\\x_3-x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

<u>Equivalently</u>

$$x_1=b_1$$

$$x_2-x_1=b_2\Rightarrow x_2=x_1+b_2=b_1+b_2$$

$$x_3-x_2=b_3\Rightarrow x_3=x_2+b_3=b_1+b_2+b_3$$

Vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}$$

linear combination with scalars b_1, b_2, b_3

$$\Rightarrow \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ or } \mathbf{x} = A^{-1}\mathbf{b}$$

↙ A^{-1}(inverse) b

(S: sum matrix)

(A^{-1} exists if A is invertible)

($A^{-1}A = I$)

($A^{-1}\mathbf{x} = \mathbf{b} \Rightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$)

(A transform $\mathbf{x} \longrightarrow \mathbf{b}$)

(A^{-1} : inverse transform $\mathbf{b} \rightarrow \mathbf{x}$)

(If $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)

(Sum matrix is the inverse of diff. matrix)

Another example (same $\mathbf{u}, \mathbf{v}$, diff. $\mathbf{w}$)

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w}^* \end{bmatrix}$$

$$\Rightarrow C\mathbf{x} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

(circular or cyclic diff. matrix)

(Recall: $A\mathbf{x} = 0 \Rightarrow \mathbf{x} = \mathbf{0}$

since $A\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x} = \mathbf{0}$ **)**

But here, $C\mathbf{x} = 0$ **has infinitely many sol. for any vector** $\mathbf{x}$ **with** $x_1 = x_2 = x_3 \Rightarrow C\mathbf{x} = \mathbf{0}$

(or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix} \Rightarrow C\mathbf{x} = \mathbf{0}$**)**

(inverse does NOT exist since cannot find C^{-1}

s.t. $C^{-1}C\mathbf{x} = C^{-1}(\mathbf{0}) = \mathbf{x}$**)**

Note that the system of equations in $C\mathbf{x} = \mathbf{b}$ is

$$\begin{aligned} x_1 - x_3 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \end{aligned}$$

Adding 3 equations together, $0 = b_1 + b_2 + b_3$

(sol. only exists when $b_1 + b_2 + b_3 = 0$)

Ex: $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \longleftrightarrow$ **no sol.**

$\Rightarrow$ **no comb, of $\mathbf{u}, \mathbf{v}, \mathbf{w}^*$ produce** $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

$\Rightarrow$ **the comb. don't fill entire 3D space or all linear comb. of $\mathbf{u}, \mathbf{v}, \mathbf{w}^*$ lie on the plane** $b_1 + b_2 + b_3 = 0$

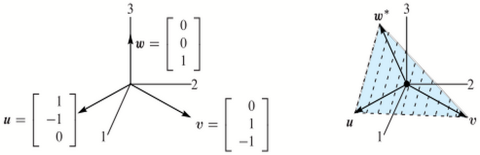

Figure 10: Independent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. Dependent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ in a plane.

u & v already forms a plane

linear indep. if w is not in the plane
linear dep. if w^* is in the plane

(Note that $\mathbf{u} + \mathbf{v} + \mathbf{w}^* = \mathbf{0}$

$$\Rightarrow \mathbf{w}^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{u} - \mathbf{v}) \quad \text{(lin. comb.!!)}$$

Fact

$\mathbf{u}, \mathbf{v}, \mathbf{w}$ are lin. indep.$\Leftrightarrow$ no comb. except

$0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w} = \mathbf{0}$ gives $\mathrm{b} = \mathbf{0}$

(lin. indep. columns : $A\mathrm{x} = 0$ has only one sol. & A is invertible)

$\mathbf{u}, \mathbf{v}, \mathbf{w}$ are lin. depend $\Leftrightarrow$ other comb. gives $\mathrm{b} = 0$

(lin. dep. col.s : $A\mathrm{x} = 0$ has many sol. & A is singular)

($\exists$ nonzero x s.t. $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = 0$

$\Rightarrow \mathbf{w} = \frac{x_1}{-x_3}\mathbf{u} + \frac{x_2}{-x_3}\mathbf{v} \Rightarrow$ cannot solve $A\mathrm{x} = \mathrm{b}, \forall \mathrm{b} \Rightarrow A$ singular)

Subspaces:

Recall: columns of C are depend.

(columns of C lie in the same plane)

(Many vectors in R^3 do not lie on that plane)

For b not in that plane, $C\mathrm{x} = \mathrm{b}$ has not sol.

Lin. comb.s of columns of C form a subspace of R^3

Recall: columns of A are indep.

$\Rightarrow$ **All comb.s of col.s of A forms the entire space**

$\Rightarrow$ $A\mathrm{x} = \mathrm{b}$ **has a sol. for every** b

$\Rightarrow$ **columns of $A\,(\mathrm{u}, \mathrm{v}, \mathrm{w})$ forms a basis for R^3**

More generally

A basis for R^n is a collection of n lin. indep. vectors in R^n

or

A comb. of n vectors whose comb. cover the entire R^3

or

A matrix has these n vectors as column vectors is invertible

Vector space

A collection of vectors are closed under lin. comb.

subsace

A vector space inside another vector space

Ex:

- **the origin**
- **a line through the origin**
- **a plane through the origin**
- **all of R^3**