EECS 205003 Session 2

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The geometry of linear equations

Central problem of linear algebra

Solving a system of linear equation

Ex

$$\mathbf{x} - 2\mathbf{y} = 1$$

$$3\mathbf{x} + 2\mathbf{y} = 11$$ (2 equations, 2 unknowns)

[We can have 3 different views on this!]

Row Picture

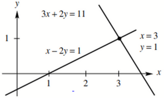

Figure 11: *Row picture*: The point (3, 1) where the lines meet is the solution.

Solution x=3,y=1 is where the two lines meet

x=3, y=1 is the point that satisfies both linear equations

Chapter 2

Column Picture

See the same line equations as vector equations

$$\Rightarrow \mathbf{x}\underset{\mathbf{c}}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} + \mathbf{y}\underset{\mathbf{d}}{\begin{bmatrix} -2 \\ 2 \end{bmatrix}} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \mathbf{b}$$

$$\Rightarrow \mathbf{xc} + \mathbf{yd} = \mathbf{b}$$

(linear combination of two column vector gives b)

Now we need to find scalars x & y

s.t x copies of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ + y copies of $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ equals the vector $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$

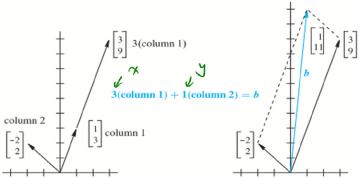

Figure 12: *Column picture*: A combination of columns produces the right side (1,11).

Chapter 2

Linear combination

$$3\begin{bmatrix}1\\3\end{bmatrix}+1\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

(same solution $\mathrm{x}=3,\mathrm{y}=1$,but different views)

Matrix Picture

$$\begin{aligned}\mathbf{x}-2\mathbf{y}&=1\\3\mathbf{x}+2\mathbf{y}&=11\end{aligned}\Rightarrow\underset{\substack{\Downarrow\\ \textbf{coefficient}\\ \textbf{matrix}}}{\overset{A}{\begin{bmatrix}1&-2\\3&2\end{bmatrix}}}\underset{\substack{\Downarrow\\ \textbf{vector of}\\ \textbf{unknown}}}{\overset{\mathbf{x}}{\begin{bmatrix}\mathbf{x}\\\mathbf{y}\end{bmatrix}}}=\overset{\mathbf{b}}{\begin{bmatrix}1\\11\end{bmatrix}}$$

$$\Rightarrow A\cdot\mathbf{x}=\mathbf{b}$$

Matrix multiplication

Method 1

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

(based on column picture)

Method 2

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \cdot 3 - 2 \cdot 1 \\ 3 \cdot 3 + 2 \cdot 1 \end{bmatrix}}_{\text{dot product}} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

(based on row picture)

Chapter 2

Three equation in three unknowns

$$Ax = \mathbf{b} \quad \Leftrightarrow \left[\begin{array}{c} x + 2y + 3z = 6 \\ 2x + 5y + 2z = 4 \\ 6x - 3y + z = 2 \end{array} \right]$$

Row Picture **(3 equations + 3 unknowns: usually one solution.)**

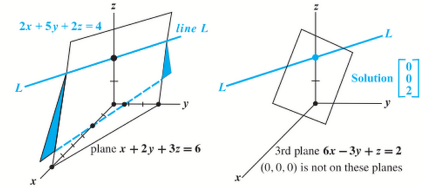

Figure 13: *Row picture*: Two planes meet at a line, three planes at a point.

Note:

x+2y+3z=0 passes through origin

x+2y+3z=6 does NOT, but is parallelly shifted

(Solution is difficult to visualize & find)

Chapter 2

Column Picture

$$x\begin{bmatrix}1\\2\\6\end{bmatrix}+y\begin{bmatrix}2\\5\\-3\end{bmatrix}+z\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}$$

(Very easy to see that x=0, y=0, z=2 in the solution)

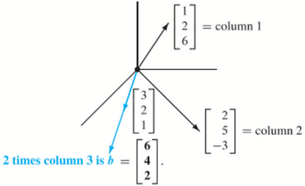

Figure 14: *Column picture*: $(x, y, z) = (0, 0, 2)$ because $2(3, 2, 1) = (6, 4, 2) = \boldsymbol{b}$.

$$\left(0\begin{bmatrix}1\\2\\6\end{bmatrix}+0\begin{bmatrix}2\\5\\-3\end{bmatrix}+2\begin{bmatrix}3\\2\\1\end{bmatrix}=\mathbf{b}\right)$$

Chapter 2

Matrix Picture

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}} \underset{\mathbf{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{\mathbf{b}}{\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}}$$

Multiplication by rows

$$A\mathbf{x} = \begin{bmatrix} (row1) \cdot \mathbf{x} \\ (row2) \cdot \mathbf{x} \\ (row3) \cdot \mathbf{x} \end{bmatrix}$$

Multiplication by columns

$$A\mathbf{x} = x(col.1) + y(col.2) + z(col.3)$$

Chapter 2

Identify Matrix

ones on the "main diagonal"

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I\mathbf{x} = \mathbf{x}$$

Matrix notation

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{bmatrix}$$

Linear independence

Q: Given a matrix A, can we solve $A\mathrm{x} = \mathrm{b}$ for every possible vector b

From column picture

Q: Do linear combination of columns of A fill the entire space? (2D OR 3D)

If not, we say A is singular

$\Rightarrow$ columns of A are linearly dependent

(For 2D, linear combination of column vectors lies on a point, or a line)

(For 3D, linear combination of column vectors lies on a point, line, or plane)

worked ex 2.1A 2.1B