EECS 205003 Session 1

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Vectors & Linear Combination

Q: Why do we need vectors?

- We cannot add apples & oranges

A column vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{matrix} \leftarrow \text{first component} \\ \leftarrow \text{second component} \end{matrix}$$

Vector addition

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

Note

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

Scalar Multiplication

$$c\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

Ex:

$$2\mathbf{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} \quad -\mathbf{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$$

Linear Combination

$c\mathbf{v} + d\mathbf{w}$

Special cases

$1\mathbf{v} + 1\mathbf{w} =$ **sum of vectors**

$1\mathbf{v} - 1\mathbf{w} =$ **difference of vectors**

$0\mathbf{v} + 0\mathbf{w} =$ **zero vector**

$$\left[\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq 0\right]$$

$c\mathbf{v} + 0\mathbf{w} =$ **vector** $c\mathbf{v}$ **in the direction of** $\mathbf{v}$

Q: How to represent vector v ?

Point in the plane

Two numbers $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ + **Arrow from** $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

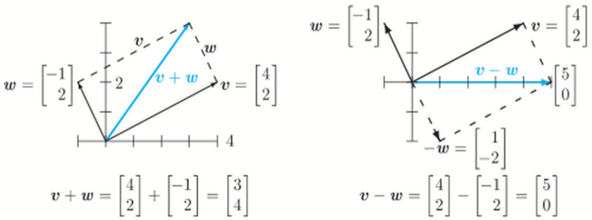


Figure 1: Vector addition $v + w = (3, 4)$ produces the diagonal of a parallelogram. The linear combination on the right is $v - w = (5, 0)$.

Vectors in 3-Dimension

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{matrix} \leftarrow \text{1st component} \\ \leftarrow \text{2nd component} \\ \leftarrow \text{3rd component} \end{matrix}$$

Visualization

column vector $\mathbf{v}$ $\leftrightarrow$ arrow from origin

$+$

points where arrow ends

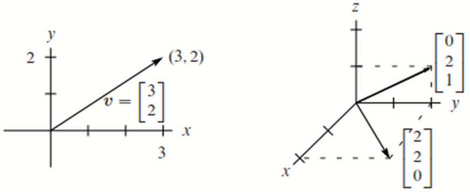


Figure 2: Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

Notation

$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ **same as v=(1,1,-1) differ from [1,1,-1]**

(column vector) **(row vector)**

$$\left(\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^{\mathrm{T}} \right)$$

Vector Addition

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \ \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \ \mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$$

Linear Combination

$c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$

Ex

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Q: What's the picture of all combinations of $c\mathbf{u}$?

The combination of $c\mathbf{u}$ fill a line
(exception: if $\mathbf{u} = 0$)

Q: What's the picture of all combinations of $c\mathbf{u} + d\mathbf{v}$?

The combination of $c\mathbf{u} + d\mathbf{v}$ fill a plane
(exception: if $\mathbf{u} = \mathbf{v} = \mathbf{0}$
or $\mathbf{u}$ and $\mathbf{v}$ in the same direction)

Q: What's the picture of all combinations of $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$?

The combination of $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ fill 3-D space

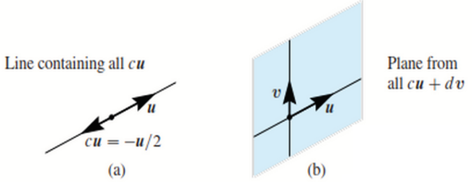


Figure 3: (a) Line through $\boldsymbol{u}$. (b) The plane containing the lines through $\boldsymbol{u}$ and $\boldsymbol{v}$.

(Always include 0, origin)

(exception: if $\mathbf{u} = \mathbf{v} = \mathbf{w} = 0$ or $\mathbf{w}$ lies on the plane of $c\mathbf{u} + d\mathbf{v}$)

Q: How about n-dim vectors ?

Can be easily generalized from the above concepts!

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \cdots$$

(For ease of visualization, focus on 2 or 3-D vectors as examples)

Length & Dot Products

$\boxed{Def}$ **The dot product or inner product of $\mathbf{u} = (u_1, u_2)$ & $\mathbf{w} = (w_1, w_2)$ is $\mathbf{u} \cdot \mathbf{w} = u_1 w_1 + u_2 w_2$**

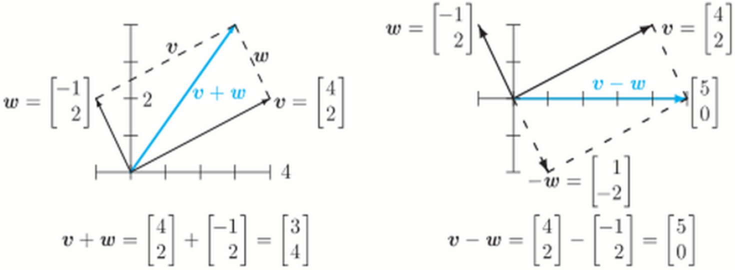


Figure 1: Vector addition $\boldsymbol{v} + \boldsymbol{w} = (3,\ 4)$ produces the diagonal of a parallelogram. The linear combination on the right is $\boldsymbol{v} - \boldsymbol{w} = (5,\ 0)$.

Q: What is the dot product of $\mathbf{u}$ & $\mathbf{w}$ in Fig 1?

$$\mathbf{u} \cdot \mathbf{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0$$

(Two vectors are perpendicular!) **(Note: $\mathbf{w} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{w}$)**

(Another easy example $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$ **)**

For general n-dim vectors

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} v_i w_i$$

Length & Unit Vector

Def **The length of a vector $\mathbf{u}$ is the square root of $\underline{\mathbf{u}} \cdot \mathbf{u}$:**

length=$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

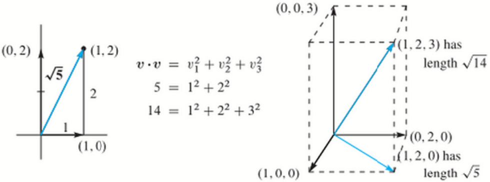


Figure 6: The length $\sqrt{v \cdot v}$ of two-dimensional and three-dimensional vectors.

Def **A unit vector is a vector of length=1, i.e.,**$\mathbf{u} \cdot \mathbf{u} = 1$

Q: How to get unit vector?

$$\mathbf{u} = \frac{\mathbf{u}}{\|\mathbf{u}\|} : \frac{(1,1,1,1)}{\sqrt{4}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$\downarrow$

(unit vector of same direction as $\mathbf{u}$)

Standard unit vectors

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$(\sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{\cos^2\theta \sin^2\theta} = 1)$

$\theta = 0^\circ \Rightarrow \mathbf{u} = \mathbf{i}, \theta = 90^\circ$ **or** $\frac{\pi}{2} \Rightarrow \mathbf{u} = \mathbf{j}$

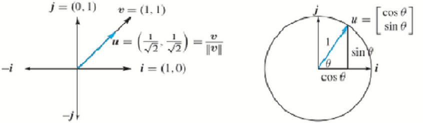
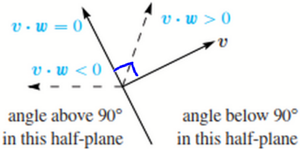


Figure 7: The coordinate vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. The unit vector $\boldsymbol{u}$ at angle 45° (left) divides $\boldsymbol{v} = (1, 1)$ by its length $\|\boldsymbol{v}\| = \sqrt{2}$. The unit vector $\boldsymbol{u} = (\cos\theta, \sin\theta)$ is at angle θ.

The Angle between two vectors



Fact **Unit vector $\mathbf{u_1} \,\&\, \mathbf{u_2}$ with angle θ in between have**

$$\mathbf{u_1} \cdot \mathbf{u_2} = \cos\theta \Rightarrow |\mathbf{u_1} \cdot \mathbf{u_2}| \leqslant 1$$

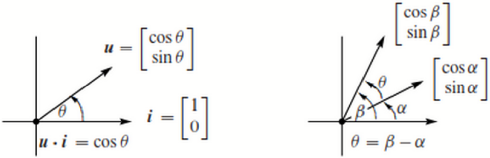


Figure 9: The dot product of unit vectors is the cosine of the angle θ.

$\boxed{Fact}$ **Cosine formula**

If $\mathbf{u}$ & $\mathbf{w}$ **are nonzero vectors then** $\frac{|\mathbf{u}\cdot\mathbf{w}|}{\|\mathbf{u}\|\|\mathbf{w}\|} = \cos\theta$

($\frac{\mathbf{u}}{\|\mathbf{u}\|}, \frac{\mathbf{w}}{\|\mathbf{w}\|}$ **are unit vector)**

$\boxed{Fact}$ **Schwarz inequality**

$$|\mathbf{u}\cdot\mathbf{w}| \le \|\mathbf{u}\|\|\mathbf{w}\|$$

(comes from $|\cos\theta| \le 1$**)** **Fact Triangle inequality**

$\boxed{Fact}$ **Triangle inequality**

$$\|\mathbf{u}+\mathbf{w}\| \le \|\mathbf{u}\| + \|\mathbf{w}\|$$