# EECS 205003 Session 30

Che Lin

Institute of Communications Engineering

Department of Electrical Engineering

## Positive definite matrices

Studying positive definite matrices brings the whole course together: pivots/ determinants/ eigenvalues/ stability

$\boxed{Def}$ A matrix is positive definite (PD)

if: 1. the matrix is symmetric

2. all $\lambda > 0$

Note: if $\lambda \geq 0$, we have a positive semidefinite matrix (PSD)

Issue: Computing eigenvalues is a lot of work!

Q: Can we have a quick test? Yes!

## Start with 2x2

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ When does $A$ have $\lambda_1 > 0,\ \lambda_2 > 0$?

Note: $\lambda$'s are real $\because$ $A$ is symmetric

$\boxed{Fact}$ The eigenvalues of $A$ are positive iff $a > 0$ & $ac - b^2 > 0$

(upper left determinant)

proof:

"$\Rightarrow$"

If $\lambda_1 > 0,\ \lambda_2 > 0$, then $ac - b^2 = det(A) = \lambda_1\lambda_2 > 0\ \Rightarrow ac > 0$

$a + c = tr(A) = \lambda_1 + \lambda_2 > 0 \Rightarrow a,\ c$ both positive

"$\Leftarrow$"

If $a > 0,\ ac - b^2 > 0$, then $c > \frac{b^2}{a} > 0$

so $\lambda_1\lambda_2 = det(A) = ac - b^2 > 0$

$\lambda_1 + \lambda_2 = tr(A) = a + c > 0 \Rightarrow \lambda_1,\ \lambda_2 > 0$

Ex:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ a > 0, \text{ but } ac - b^2 = 1 - 4 < 0 \ \textbf{(x)}$$

$$A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix},\ a > 0, \text{ but } ac - b^2 = 6 - 4 > 0 \ \textbf{(v)}$$

$$A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix},\ ac - b^2 = 6 - 4 > 0, \text{ but } a < 0 \ \textbf{(x)}$$

$\boxed{Fact}$ The eigenvalues of $A = A^T$ are positive iff the pivots are positive, i.e., $a > 0$ & $\frac{ac-b^2}{a} > 0$

proof: recall for symmetric matrices
number of positive eigenvalues = number of positive pivots

check the pivots:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & c - \frac{b}{a}b \end{bmatrix}$$ pivots: $a, \; c - \frac{b^2}{a} = \frac{ac-b^2}{a}$

(To determine a PD matrix, this is a lot faster than computing eigenvalues!)

Back to example:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix}$$

pivots: 1 & -3 | 1 & 2 | -1 & -2

**(indefinite)** | **(positive definite)** | **(negative definite)**

## Energy-based definition

If $\lambda$'s $> 0$ from $A\mathbf{x} = \lambda\mathbf{x}$

$\mathbf{x}^{\mathbf{T}}A\mathbf{x} = \lambda\mathbf{x}^{\mathbf{T}}\mathbf{x} = \lambda \parallel \mathbf{x} \parallel^2 > 0$ (true for any eigenvectors)

New idea: Not just for eigenvectors but $\forall$ nonzero vectors $\mathbf{x}$

$\mathbf{x}^{\mathbf{T}}A\mathbf{x} > 0$ (energy of the system)

$\boxed{Def}$ (The common definition of PD)

$A$ is PD if $\mathbf{x}^{\mathbf{T}}A\mathbf{x} > 0$ for all nonzero $\mathbf{x}$:

$$\mathbf{x}^{\mathbf{T}}A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 > 0$$

($2bxy$ from off-diagonal $b$ & $b$) ( $ax^2, \ cy^2$ from diagonal $a, \ c$)

# 6.5 Positive Definite Matrices

$\boxed{Fact}$ If $A,\ B$ are PD, so is A+B

proof: $\mathbf{x^T}(A+B)\mathbf{x} = \mathbf{x^T}A\mathbf{x} + \mathbf{x^T}B\mathbf{x} > 0$

$\Rightarrow\ A+B$ is PD

(pivots & eigenvalues are not easy to follow when matrices are added, but the energies just add!)

$\boxed{Fact}$ If the columns of R are independent then $A = R^T R$ is PD

(R can be rectangular, but $A = R^T R$ is square & symmetric)

proof: $\mathbf{x^T}A\mathbf{x} = \mathbf{x^T}R^T R\mathbf{x} = (R\mathbf{x})^T(R\mathbf{x}) = \| R\mathbf{x} \|^2$

$R\mathbf{x} \neq 0$ when $\mathbf{x} \neq 0$ if columns of $R$ are independent ( no free columns)$\Rightarrow \mathbf{x^T}A\mathbf{x} > 0$

$\Rightarrow A$ is PD

# 6.5 Positive Definite Matrices

*Statements* (Five equivalent statements of PD)

When a symmetric matrix is PD , the following statements are equivalent:

1. All n pivots $> 0$
2. All upper left determinant $> 0$
3. All n eigenvalues $> 0$
4. $\mathbf{x}^{\mathbf{T}} A\mathbf{x} > 0$ except at $\mathbf{x} = 0$ (energy-based definition)
5. $A = R^T R$ & $R$ has independent columns

Q: How to link 1-3 with 4-5 ? Show by an example (almost a proof)

Ex: Test $A$ & $B$ for PD

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

For $A$:

1. pivots: $2, \frac{3}{2}, \frac{4}{3}$ (multiplier: $-\frac{1}{2}, -\frac{2}{3}$)
2. upper left determinant: $2, 3, 4$
3. eigenvalues: $2-\sqrt{2}, 2+\sqrt{2}$
4. $\mathbf{x}^{\mathbf{T}}A\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= 2({x_1}^2 - x_1x_2 + {x_2}^2 - x_2x_3 + {x_3}^2)$

$= 2(x_1 - \frac{1}{2}x_2)^2 + \frac{3}{2}(x_2 - \frac{2}{3}x_3)^2 + \frac{4}{3}{x_3}^2 > 0$, if all pivots $> 0$

(green: pivots, blue: multipliers)

Q: Is this a coincidence? No, we will see why later

5. $A = R^T R$

choice one:

$A = LDL^T$ (symmetric version of $LU$ decomposition)

$\Rightarrow \; A = LDL^T = (L\sqrt{D})(L\sqrt{D})^T = R^T R$ (cholesky factor)

$\Rightarrow \; A$ is PD since $L^T$ has independent columns. Specifically,

$$A = LDL^T = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{x^T} A \mathbf{x} = \mathbf{x^T} L \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \mathbf{x^T} L \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} x_1 - \frac{1}{2}x_2 \\ x_2 - \frac{2}{3}x_3 \\ x_3 \end{bmatrix}$$

$$= 2(x_1 - \tfrac{1}{2}x_2)^2 + \tfrac{3}{2}(x_2 - \tfrac{2}{3}x_3)^2 + \tfrac{4}{3}{x_3}^2$$

choice two:

$$A = R^T R, \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

(first difference matrix)

For B: Determinant test is easiest

$\Rightarrow$ only need to check

$det(B) = 4 + 2b - 2b^2 = (1 + b)(4 - 2b)$

At $b = -1$ or $b = 2,\ det(B) = 0$

$-1 < b < 2 \ \Rightarrow \ det(B) > 0 \Rightarrow B$ is PD

## Positive semidefinite matrices(PSD)

At the edge of PD, $\mathbf{x}^{\mathbf{T}}A\mathbf{x} \geq 0$ or smallest eigenvalues $= 0$ or $det = 0$

Ex: $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, $det(A) = 0$

$A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ eigenvector

$\mathbf{x}^{\mathbf{T}}A\mathbf{x} = 0$ for this eigenvector

$\mathbf{x}^{\mathbf{T}}A\mathbf{x} > 0$ for all other directions

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(cyclic $A$ from cyclic $R$) $\downarrow$ dependent columns