EECS 205003 Session 28

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Complex matrices

Matrices with all real entries can still have complex eigenvalues

$\Rightarrow$ We cannot avoid dealing with complex numbers !

Complex vectors

Length:

Given a vector $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$

with complex entries

Q: How do we find its length?

Our old definition:

$$\mathbf{z}^T\mathbf{z} = \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$ (No good!) (Not always positive!)

Ex: $\mathbf{z} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$\mathbf{z}^H\mathbf{z} = \begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 0 \text{ ?}$$

Correct definition:

$$\|\mathbf{z}\|^2 = \bar{\mathbf{z}}^T\mathbf{z} = \begin{bmatrix} \bar{z_1} & \cdots & \bar{z_n} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

$$= |z_1|^2 + \cdots + |z_n|^2 \geq 0$$

Ex:

$$(\text{length}\begin{bmatrix}1\\ i\end{bmatrix})^2 = \begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix} = 2 \text{ (v)}$$

simplify notation:

$\|\mathbf{z}\|^2 = \mathbf{z}^H\mathbf{z}$ where $\mathbf{z}^H = \bar{\mathbf{z}}^T$

Same for matrices:

If $A = \begin{bmatrix}1 & i\\ 0 & 1+i\end{bmatrix}$, $A^H = \begin{bmatrix}1 & 0\\ -i & 1-i\end{bmatrix}$

Inner product

$$\mathbf{y}^H\mathbf{x} = \bar{\mathbf{y}}^T\mathbf{x} = \bar{y_1}x_1 + \cdots + \bar{y_n}x_n$$

Note:

$$\mathbf{y}^H\mathbf{x} \neq \mathbf{x}^H\mathbf{y} = \bar{x_1}y_1 + \cdots + \bar{x_n}y_n$$

$$= \text{complex conjugate of } \mathbf{y}^H\mathbf{x}$$

(order is important !)

Ex: $\mathbf{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$$\mathbf{u}^H \mathbf{v} = \begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = 0 \text{ (orthogonal)}$$

Note: $(A\mathbf{u})^H \mathbf{v} = \mathbf{u}^H (A^H \mathbf{v})$

Reason: $(A\mathbf{u})^H = \overline{A\mathbf{u}}^T = \bar{\mathbf{u}}^T \bar{A}^T = \mathbf{u}^H A^H$

(Inner product of $A\mathbf{u}$ with $\mathbf{v}$ equals Inner product of $\mathbf{u}$ with $A^H \mathbf{v}$)

Note: $(AB)^H = B^H A^H$

Hermitian matrices

Recall: For symmetric matrix $A = A^T$

$\Rightarrow$ real eigenvalues

$\Rightarrow$ there is a full set of orthogonal eigenvectors

$\Rightarrow$ Diagonalizing matrix $S = Q$ (orthogonal)

$\Rightarrow A = Q\Lambda Q^{-1}$ or $A = Q\Lambda Q^T$

(All this follows from $a_{ij} = a_{ji}$ when A is real)

Now for complex matrices

We have Hermitian matrix $A = A^H$ where $a_{ij} = \overline{a_{ji}}$

Note: Every symmetric matrix is Hermitian

$(a_{ij} = a_{ji} = \overline{a_{ji}}$ for real $a_{ji})$

Ex: Hermitian matrix

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} = A^H$$

Fact: If $A = A^H$ and $\mathbf{z}$ is any vector then $\mathbf{z}^H A\mathbf{z}$ is real

proof: $\mathbf{z}^H A\mathbf{z}$ is 1×1 number

$$\Rightarrow (\mathbf{z}^H A\mathbf{z})^H = \mathbf{z}^H A^H (\mathbf{z}^H)^H = \mathbf{z}^H A\mathbf{z}$$

the number is real since it is equal to its conjugate

Back to example:

$$\begin{bmatrix} \bar{z_1} & \bar{z_2} \end{bmatrix} \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \underbrace{2\bar{z_1}z_1 + 5\bar{z_2}z_2}_{\text{(diagonal)}} + \underbrace{(3-3i)\bar{z_1}z_2 + (3+3i)z_1\bar{z_2}}_{\text{(off-diagonal)}}$$

($2|z_1|^2$ & $5|z_2|^2$ are both real and the off-diagonal terms are conjugate of each other $\Rightarrow$ sum is real)

Fact: Every eigenvalue of a Hermitian matrix is real

proof: Suppose $A\mathbf{z} = \lambda\mathbf{z}$

$$\Rightarrow \underbrace{\mathbf{z}^H A\mathbf{z}}_{\text{real}} = \lambda\mathbf{z}^H\mathbf{z} = \lambda \underbrace{|z|^2}_{\text{real}}$$

so λ must be real !

Back to example:

$$\begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 - |3+3i|^2$$
$$= \lambda^2 - 7\lambda + 10 - 18$$
$$= (\lambda - 8)(\lambda + 1)$$

$\Rightarrow \lambda = 8 \ \& \ -1$

Fact The eigenvectors of a Hermitian matrix are orthogonal (when they correspond to different eigenvalues)

If $A\mathbf{z} = \lambda\mathbf{z}$ & $A\mathbf{y} = \beta\mathbf{y}$ & $\lambda \neq \beta$ then $\mathbf{y}^{\mathbf{H}}\mathbf{z} = 0$

proof:

$$A\mathbf{z} = \lambda\mathbf{z} \Rightarrow \mathbf{y}^H A\mathbf{z} = \lambda\mathbf{y}^H\mathbf{z}$$
$$\mathbf{y}^H A^H = \beta\mathbf{y}^H \Rightarrow \mathbf{y}^H A^H\mathbf{z} = \beta\mathbf{y}^H\mathbf{z}$$
$$\Rightarrow (\lambda - \beta)\mathbf{y}^H\mathbf{z} = 0 \Rightarrow \mathbf{y}^H\mathbf{z} = 0 \text{ if } \lambda \neq \beta$$

Back to example:

$$(A-8I)\mathbf{z} = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{z} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$(A+I)\mathbf{y} = \begin{bmatrix} 3 & 3-3i \\ 3+3i & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{y} = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$$

$$\Rightarrow \mathbf{y}^H\mathbf{z} = \begin{bmatrix} 1+i & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = 0$$

Note: Eigenvectors have length $\sqrt{3}$. After dividing by $\sqrt{3}$, they are orthonormal

⇒They go into eigenvector matrix S that diagonalize A

(When A is real & symmetric, S is Q-orthogonal. When A is complex & Hermitian eigenvectors are complex & orthonormal

⇒ S is like Q but complex)

(Complex & orthogonal ⇒ unitary)

Unitary matrices

A unitary matrix U is a complex square matrix that has orthonormal columns

(U is a complex equivalent of Q)

Ex: Eigenvector matrix of A

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

Recall: For orthonormal matrix Q (real), $Q^T Q = I$

Q: What does it mean for complex vectors $\mathbf{q_1}, \cdots, \mathbf{q_n}$ to be orthonormal ?

Use new definition of inner product

$$\Rightarrow \mathbf{q_j^H q_k} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$Q = \begin{bmatrix} \mathbf{q_1} & \cdots & \mathbf{q_n} \end{bmatrix} \Rightarrow Q^H Q = I$$

Fact: Every matrix U with orthonormal columns has $U^H U = I$

If U is square, then $U^H = U^{-1}$

Fact: If U is unitary, then $\|Uz\| = \|z\|$

$\Rightarrow U\mathbf{z} = \lambda \mathbf{z}$ leads to $|\lambda| = 1$

proof: $\|U\mathbf{z}\|^2 = \mathbf{z}^H U^H U \mathbf{z} = \mathbf{z}^H \mathbf{z} = \|\mathbf{z}\|^2$

Back to example:

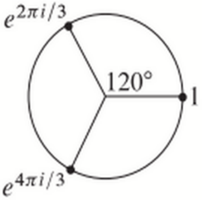
$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$ both Hermitian & unitary

$\Rightarrow$ real eigenvalues & $|\lambda| = 1$

$\Rightarrow \lambda = 1$ or -1

since trace $= 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -1$

Ex: 3×3 Fourier matrix



Fourier matrix $F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}$

Figure 61: The cube roots of 1 go into the Fourier matrix $F = F_3$.

Q: Is it Hermitian ?

$$F^H = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{-2\pi i}{3}} & e^{\frac{-4\pi i}{3}} \\ 1 & e^{\frac{-4\pi i}{3}} & e^{\frac{-2\pi i}{3}} \end{bmatrix} \neq F$$

Q: Is it unitary ?

The squared length of each column $= \frac{1}{3}(1+1+1) = 1$ (unit vectors)

$$\begin{aligned}(col.1)^H(col.2) &= \frac{1}{3}(1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}}) \\ &= 0 \\ (col.2)^H(col.3) &= \frac{1}{3}(1 \cdot 1 + e^{\frac{-2\pi i}{3}} e^{\frac{4\pi i}{3}} + e^{\frac{-4\pi i}{3}} e^{\frac{2\pi i}{3}}) \\ &= \frac{1}{3}(1 + e^{\frac{2\pi i}{3}} + e^{\frac{-2\pi i}{3}}) \\ &= 0\end{aligned}$$

$\Rightarrow F$ is unitary !

(Read real v.s. complex p.506)