EECS 205003 Session 27

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Symmetric matrices

Recall: A is symmetric if $A^T = A$

If a matrix has special properties(e.q., Markov matrices), its eigenvalues & eigenvectors are likely to have special properties

Q: What is special about $A\mathbf{x} = \lambda\mathbf{x}$ if A is symmetric?

Fact For a symmetric matrix with real entries, we have

1. All eigenvalues are real
2. Eigenvectors can be chosen to be orthonormal

Note: Every symmetric matrix can be diagonalized (will prove this later when repeated eigenvalues)

Note: Its eigenvector matrix S becomes an orthogonal matrix Q where $Q^{-1} = Q^T$

This leads to the Spectral Theorem

Spectral Theorem Every symmetric matrix has the factorization $A = Q\Lambda Q^T$ with real eigenvalues in Λ and orthonormal eigenvectors in $S = Q$

Note: Easy to see $Q\Lambda Q^T$ is symmetric. Any $A = Q\Lambda Q^T$ is symmetric

Note: This is "Spectral Theorem" in math & "Principal Axis Theorem" in mechanics and physics

Reason: Approach in 3 steps

Step 1: By an example, showing real λ's in Λ & orthonormal $\mathbf{x}$ in Q

Step 2: By a proof when no repeated eigenvalues

Step 3: By a proof that allows repeated eigenvalues

Ex.1 (p.331) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|A - \lambda I| = \lambda^2 - 5\lambda \Rightarrow \lambda = 0$ or 5

(can see this directly: A is singular

$\Rightarrow \lambda_1 = 0$ is an eigenvalue. $tr(A) = 1 + 4 = 5$

$\Rightarrow \lambda_1 + \lambda_2 = 5 \Rightarrow \lambda_2 = 5$)

Eigenvectors:

$A\mathbf{x_1} = 0 \Rightarrow \mathbf{x_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$(A - 5I)\mathbf{x_2} = \mathbf{0} \Rightarrow \mathbf{x_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q: Why $\mathbf{x_1}$ & $\mathbf{x_2}$ are orthogonal?

$\mathbf{x_1}$ in $N(A)$, $\mathbf{x_2}$ in $C(A)$

($A\mathbf{x_2} = 5\mathbf{x_2} \Rightarrow \mathbf{x_2}$ is a combination of columns of $A \Rightarrow \mathbf{x_2} \in C(A)$)

Q: $N(A) \perp C(A^T)$ not $C(A)$?

But A is symmetric $\Rightarrow A^T = A$

$\Rightarrow C(A^T) = C(A)$ (row space = column space)

Normalize $\mathbf{x_1}$ & $\mathbf{x_2}$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = Q\Lambda Q^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} (\vee)$$

Fact All eigenvalues of real symmetric matrix are real

proof: $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow \bar{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$

$(\lambda = a + ib,\ \bar{\lambda} = a - ib)$

$\Rightarrow A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$ (A is real)

(complex eigenvalues of real A always comes in conjugate pairs)

Take transpose

$\Rightarrow \bar{\mathbf{x}}^T A^T = \bar{\mathbf{x}}^T \bar{\lambda}$

$\Rightarrow \bar{\mathbf{x}}^T A = \bar{\mathbf{x}}^T \bar{\lambda}$ $(A = A^T)$

Multiply by $\mathbf{x}$ on the right

$\Rightarrow \bar{\mathbf{x}}^T A\mathbf{x} = \bar{\mathbf{x}}^T \bar{\lambda}\mathbf{x}$ ——— ①

On the other hand, $A\mathbf{x} = \lambda\mathbf{x}$

Multiply by $\bar{\mathbf{x}}^T$ on the left

$\Rightarrow \bar{\mathbf{x}}^T A\mathbf{x} = \bar{\mathbf{x}}^T \lambda\mathbf{x}$ ——— ②

Comparing ① & ②

$\Rightarrow \lambda\bar{\mathbf{x}}^T\mathbf{x} = \bar{\lambda}\bar{\mathbf{x}}^T\mathbf{x}$

$\Rightarrow \lambda = \bar{\lambda}$ if $\bar{\mathbf{x}}^T\mathbf{x} \neq 0$

$$(\bar{\mathbf{x}}^T\mathbf{x} = \begin{bmatrix} \bar{x_1} & \cdots & \bar{x_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = |x_1|^2 + \cdots + |x_n|^2$$

$\Rightarrow \bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x} \neq 0$ if $\mathbf{x} \neq \mathbf{0}$)

Note: eigenvectors come from solving $(A - \lambda I)\mathbf{x} = 0$ since λ all real

$\Rightarrow$eigenvectors all real

Fact Eigenvectors of a real symmetric matrix (correspond to different eigenvalues) are always perpendicular

proof:

Let $A\mathbf{x} = \lambda_1\mathbf{x}$, $A\mathbf{y} = \lambda_2\mathbf{y}$

$\Rightarrow (A\mathbf{x})^T\mathbf{y} = \lambda_1\mathbf{x}^{\mathbf{T}}\mathbf{y}$, $\mathbf{x}^{\mathbf{T}}A\mathbf{y} = \lambda_2\mathbf{x}^{\mathbf{T}}\mathbf{y}$

$\Rightarrow \mathbf{x}^{\mathbf{T}}A^T\mathbf{y} = \lambda_1\mathbf{x}^{\mathbf{T}}\mathbf{y}$

$\|$ $\Rightarrow \mathbf{x}^{\mathbf{T}}\mathbf{y} = 0$ $(\lambda_1 \neq \lambda_2)$

$\mathbf{x}^{\mathbf{T}}A\mathbf{y} = \lambda_2\mathbf{x}^{\mathbf{T}}\mathbf{y}$

$\Rightarrow$eigenvector for $\lambda_1 \perp$

eigenvector for λ_2

(True for any pair)

Note: If A has complex entries, A has real eigenvalues & perpendicular eigenvectors iff $A = \bar{A}^T$

(Proof of this follows same pattern)

Projection onto eigenvectors

If $A = A^T$, we have

$$A = Q\Lambda Q^T$$

$$= \begin{bmatrix} \mathbf{q_1} & \cdots & \mathbf{q_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{q_1^T} \\ \vdots \\ \mathbf{q_n^T} \end{bmatrix}$$

$$= \lambda_1 \mathbf{q_1} \mathbf{q_1^T} + \cdots + \lambda_n \mathbf{q_n} \mathbf{q_n^T}$$

$$= \lambda_1 P_1 + \cdots + \lambda_n P_n$$

$\downarrow$ $\downarrow$

(projection onto eigenvectors)

(Every symmetric matrix is a combination of perpendicular projection matrices)

Eigenvalues v.s. Pivots

For eigenvalues, we solve $\det(A - \lambda I) = 0$

For pivots, we use Elimination (very different !)

Only connection so far:

product of pivots = determinant

=products of eigenvalues

For symmetric matrices,

# of positive eigenvalues = # of positive pivots

Special case: A has all $\lambda_i > 0$ iff all pivots are positive

(see Ex.4 on p.334 for a sketch of proof)

Note: For large matrix, it is impractical to compute $|A - \lambda I| = 0$

But NOT hard to compute pivots by elimination

$\Rightarrow$ can use signs of pivots to determine signs of λ

e.q., eigenvalues of $A - bI$ are b less than eigenvalues of A,

check pivots > 0 or < 0

$\Rightarrow \lambda - b > 0$ or < 0

$\Rightarrow \lambda > b$ or $\lambda < b$

(we can check whether $\lambda > b$ or $\lambda < b$ for any b !)

Now, we try to show that even for repeated eigenvalues, $A = A^T$ has perpendicular eigenvectors

Fact Every square matrix factors into $A = QTQ^{-1}$

where T: upper triangular, $\bar{Q}^T = Q^{-1}$

If A has real eigenvalues, then Q & T can be chosen to be real: $Q^T Q = I$

Fact Eigenvectors of a real symmetric matrix (even with repeated eigenvalues) are always perpendicular

Proof: For symmetric matrix A, eigenvalues are all real

$\Rightarrow A = QTQ^T$, $Q^TQ = I$

$\Rightarrow Q^TAQ = Q^TQTQ^TQ = T$

since $A = A^T \Rightarrow T = T^T$ but T is upper-triangular

$\Rightarrow T = \Lambda$ is diagonal

$\Rightarrow A = Q\Lambda Q^T$ for orthogonal Q

$\Rightarrow A$ has orthonormal eigenvectors