EECS 205003 Session 26

Che Lin

Institute of Communications Engineering

Department of Electrical Engineering

Markov matrices ; Fourier series

Markov matrices

Suppose we have a positive vector $\mathbf{u_0} = \begin{bmatrix} a \\ 1-a \end{bmatrix}$

$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ (Markov matrix columns add to 1)

if $\mathbf{u_1} = A\mathbf{u_0}$, $\mathbf{u_2} = A\mathbf{u_1} \cdots$

Q: What happens if we keep doing this?

$\mathbf{u_1}, \mathbf{u_2}, \cdots$ converges to $\mathbf{u}_\infty$ (steady state)

For $\mathbf{u}_\infty$, $\mathbf{u}_\infty = A\mathbf{u}_\infty$

(multiplied by A does NOT change $\mathbf{u}_\infty$)

($\mathbf{u}_\infty$ is an eigenvector with $\lambda = 1$)

Def Markov matrix

A is a Markov matrix if:

1. Every entry of A is nonnegative
2. Every col. of A adds to 1

Fact

1. For nonnegative $\mathbf{u_0}$, $\mathbf{u_1} = A\mathbf{u_0}$ is also nonnegative
2. If components of $\mathbf{u_0}$ add to 1, so do the components of $\mathbf{u_1} = A\mathbf{u_0}$

Reason:

1. trivial since both A & $\mathbf{u_0}$ are nonnegative
2. components of $\mathbf{u_0}$ add to 1

$\Rightarrow [1, \cdots, 1]\ \mathbf{u_0} = 1$

A is Markov $\Rightarrow$ every column of A adds to 1 $\Rightarrow [1, \cdots, 1]A = [1, \cdots, 1]$

$$\underbrace{[1,\cdots,1]\ A\ \mathbf{u_0}}_{(=\mathbf{u_1})} = [1,\cdots,1]\ \mathbf{u_0} = 1$$

$\Rightarrow$ components of $\mathbf{u_1}$ add to 1

Note: same fact applies to

$\mathbf{u_2} = A\mathbf{u_1}$, $\mathbf{u_3} = A\mathbf{u_2}$, $\cdots$

$\Rightarrow$ every $\mathbf{u_k} = A^k\mathbf{u_0}$ is nonnegative with components adding to 1

($\mathbf{u_1}$, $\mathbf{u_2}$, $\cdots$, $\mathbf{u_k}$, $\cdots$ are probability vectors. The limit $\mathbf{u_\infty}$ is also a probability vector but we have to show that such limit exists)

Note: A^k is also a Markov matrix

$([1, \cdots, 1]A^k = [1, \cdots, 1]AA^{k-1} = [1, \cdots, 1]A^{k-1}$
$= \cdots = [1, \cdots, 1]A = 1)$

Ex.1 (p.432)

Fraction of rental cars in Denver starts at 0.02 (outside is 0.98)

Every month : 80% of Denver cars stay in Denver (20% leave),

5% of outside cars comes in (95% stay outside)

$$\Rightarrow \begin{bmatrix} u_{Denver} \\ u_{outside} \end{bmatrix}_{t=k+1} = \underset{A}{\begin{bmatrix} 0.8 & 0.05 \\ 0.2 & 0.95 \end{bmatrix}} \begin{bmatrix} u_{Denver} \\ u_{outside} \end{bmatrix}_{t=k}$$

$$\begin{bmatrix} u_{Denver} \\ u_{outside} \end{bmatrix}_{t=0} = \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{Denver} \\ u_{outside} \end{bmatrix}_{t=1} = \begin{bmatrix} 0.8 & 0.05 \\ 0.2 & 0.95 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 0.065 \\ 0.935 \end{bmatrix}$$

Q: What happens in the long run?

We are studying equations : $\mathbf{u_{k+1}} = A\mathbf{u_k}$

$\Rightarrow \mathbf{u_k} = A^k\mathbf{u_0} = c_1\lambda_1^k\mathbf{x_1} + \cdots + c_n\lambda_n^k\mathbf{x_n}$

Need eigenvalues & eigenvectors to diagonalize A

$|A - \lambda I| = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 0.75$

$\Rightarrow (A - I)\mathbf{x_1} = \mathbf{0} \Rightarrow \mathbf{x_1} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$ (components add to 1)

$(A - 0.75I)\mathbf{x_2} = \mathbf{0} \Rightarrow \mathbf{x_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\mathbf{u_0} = \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix} = c_1\mathbf{x_1} + c_2\mathbf{x_2} = 1\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} + 0.18\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\Rightarrow \mathbf{u_k} = 1(1)^k\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} + 0.18(0.75)^k\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(as k$\rightarrow \infty$) (steady state) (vanishing)

(eigenvector with $\lambda = 1$ is the steady-state)

(other eigenvector $\mathbf{x_2}$ disappears $\because |\lambda| < 1$)

(More steps we take, closer to $\mathbf{u}_\infty = (0.2, 0.8)$)

(True even when $\mathbf{u_0} = (0, 1)$)

Fact A is a positive Markov matrix, then $\lambda_1 = 1$ is larger than
$(a_{ij} > 0)$

any other eigenvalues. The eigenvector $\mathbf{x_1}$ is the steady-state

$\mathbf{u_k} = \mathbf{x_1} + c_2(\lambda_2)^k\mathbf{x_2} + \cdots + c_n(\lambda_n)^k\mathbf{x_n}$

$\mathbf{u}_\infty = \mathbf{x_1}$ for any initial $\mathbf{u_0}$

Reason:

1. $\lambda = 1$ is an eigenvalue:

   Every column of $A - I$ adds to $1 - 1 = 0$

   $\Rightarrow$ rows of $A - I$ add to the zero row

   $\Rightarrow A - I$ is singular

   $\Rightarrow |A - I| = 0 \Rightarrow \lambda = 1$

Alternative reason:

rows of $A - I$ add to the zero row

$\Rightarrow$[1,$\cdots$,1]$(A - I)$ = [0,$\cdots$,0]

$$\Rightarrow (A^T - I) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{0}$$

$\Rightarrow \lambda = 1$ is an eigenvalue of A^T

$\Rightarrow \lambda = 1$ is an eigenvalue of A

($|A - \lambda I| = \left|(A - \lambda I)^T\right| = \left|A^T - \lambda I\right| \Rightarrow A$ & A^T have same **eigenvalues)**

2. No eigenvalue can have $|\lambda| > 1$:

If there is any eigenvalue $|\lambda| > 1$

$\Rightarrow A^k$ will grow

But A^k is a Markov matrix

$\Rightarrow$ every column of A^k adds to 1

$\Rightarrow$ no room to grow $\Rightarrow$ contradiction !

3. $c_1 = 1$ if components of $\mathbf{u_0}$ & $\mathbf{x_1}$ add to 1:

   $[1,\cdots,1]A\mathbf{x_i}=[1,\cdots,1]\lambda_i\mathbf{x_i}$

   $[1,\cdots,1]\mathbf{x_i}=\lambda_i[1,\cdots,1]\mathbf{x_i}$

   For $\lambda_i, i \geq 2, \lambda_i \neq 1 \Rightarrow [1,\cdots,1]\mathbf{x_i} = 0$

   $\mathbf{u_0} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_n\mathbf{x_n}$

   $\Rightarrow[1,\cdots,1]\mathbf{u_0} = c_1[1,\cdots,1]\mathbf{x_1}$

   $\Rightarrow c_1 = 1$ if components of $\mathbf{u_0}$ & $\mathbf{x_1}$ add to 1

Note: In some applications, Markov matrices are defined differently: rows add up to 1 instead (calculations are transpose of everything we've done here)

Fourier series & projections

Expansion with an orthonormal basis

If we have an orthonormal basis $\mathbf{q_1}$, $\mathbf{q_2}$, $\cdots$, $\mathbf{q_n}$, we can write any vector as

$$\mathbf{v} = x_1\mathbf{q_1} + \cdots + x_n\mathbf{q_n}$$

where $\mathbf{q_i^T v} = x_1\mathbf{q_i^T q_1} + \cdots + x_i + \cdots + x_n\mathbf{q_i^T q_n}$
$= x_i \ (\mathbf{q_i^T q_j} = \mathbf{0},\ i \neq j)$

In terms of matrix:

$$[\mathbf{q_1}, \cdots, \mathbf{q_n}] \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix} = \mathbf{v}$$

$\Rightarrow Q\mathbf{x} = \mathbf{v} \Rightarrow \mathbf{x} = Q^{-1}\mathbf{v} = Q^T\mathbf{v}$

$\Rightarrow x_i = \mathbf{q_i^T v}$

(key idea: express $\mathbf{v}$ = combination of projection onto orthonormal basis vectors)

Fourier series

Same idea on functions !

$f(x) = a_0 + a_1 cosx + b_1 sinx + a_2 cos2x + b_2 sin2x + \cdots$

(express $f(x)$ as combination of projection onto trigonometric fuctions)

(extend to inifinte series)

vectors: functions

basis: 1. $cosx, sinx, cos2x, sin2x, \cdots$

Q: What does orthogonal mean in this context?

Need to define inner product first

Vectors in R^n:

$$\mathbf{v}^\mathbf{T}\mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

For functions:

$$(f.q) = \int_0^{2\pi} f(x)g(x)dx$$

(integrate over [0, 2π] since Fourier series are periodic, i.e., $f(x) = f(x + 2\pi)$)

Check orthogonality:

$$\int_0^{2\pi} sinxcosxdx = \frac{1}{2}(sinx)^2|_0^{2\pi} = 0$$

$\vdots$ (inner product=0)

Q: How to find Fourier coefficient $a_0, a_1, b_1, \cdots$?

a_0: average of $f(x)$

$$\left(\frac{1}{2\pi}\int_0^{2\pi} f(x)dx = a_0 + \frac{1}{2\pi}\int_0^{2\pi} a_1 cosxdx + \frac{1}{2\pi}\int_0^{2\pi} b_1 sinxdx + \cdots = a_0\right)$$

$a_1 : \int_0^{2\pi} f(x)cosxdx$

$= \int_0^{2\pi}(a_0 + a_1cosx + b_1sinx + \cdots)cosxdx$

$= 0 + \int_0^{2\pi} a_1cos^2xdx + 0 + \cdots$

$= \int_0^{2\pi} a_1\frac{1+cos2x}{2}dx = \pi a_1$

$\Rightarrow a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)cosxdx$

Similarly,

$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)coskxdx$

$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)sinkxdx$

(read Ex.3, p.449)