EECS 205003 Session 25

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Differential equations & e^{At}

Scalar ODE (one equation)

$\frac{du}{dt} = \lambda u$ **has sol.s u(t)=**$ce^{\lambda t}$

at $t = 0$, $u(0) = c$

$\Rightarrow u(t) = u(0)e^{\lambda t}$

Q:How about n equations?

Start with 2 equations

$\frac{du_1}{dt} = -u_1 + 2u_2$ **describe how values of var.s** $u_1 \& u_2$ **affect**

$\frac{du_2}{dt} = u_1 - 2u_2$ **each other over time**

Just as we apply linear algebra to solve difference equations, we can use it to solve differential equations

Differential equations: $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ starting from $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

We can guess that $\mathbf{u} = e^{\lambda t}\mathbf{x}$ is a sol. when $A\mathbf{x} = \lambda\mathbf{x}$

(eigenvalue & eigenvectors)

Q: Is this true ?

$\frac{d\mathbf{u}}{dt} = \lambda e^{\lambda t}\mathbf{x}$

$A\mathbf{u} = e^{\lambda t}A\mathbf{x} = \lambda e^{\lambda t}\mathbf{x}$

$\Rightarrow \frac{d\mathbf{u}}{dt} = A\mathbf{u}$

Back to example:

$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$: **singular** $\Rightarrow \lambda_1 = 0$

trace(A) **= -3 =** $\lambda_1 + \lambda_2 \Rightarrow \lambda_2 = -3$

Find corresponding eigenvectors :

$$A\mathbf{x_1} = \mathbf{0} \Rightarrow \mathbf{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A+3I)\mathbf{x_2} = \mathbf{0} \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x_2} = \mathbf{0}$$

$$\Rightarrow \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \mathbf{u_1}(t) = e^{\lambda_1 t}\mathbf{x_1} = e^{0t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{u_2}(t) = e^{\lambda_2 t}\mathbf{x_2} = e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ **(pure solutions)**

Complete sol. :

$$\mathbf{u}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(steady state solution) (decays to zero as t $\rightarrow \infty$)

$$\mathbf{u}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = c_2 = \tfrac{1}{3}$$

$$\mathbf{u}(t) = \tfrac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \tfrac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\mathbf{u}(\infty) = \tfrac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **(steady state solution)**

Summary:

Step 1: **Write u**(0) **as combination** $c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}$ **of eigenvectors of** A

Step 2: **Multiply each eigenvector** $\mathbf{x_i}$ **by** $e^{\lambda i t}$ **(pure solution)**

Step 3: **Complete solution is a combination of pure solutions**

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x_1} + \cdots + c_n e^{\lambda_n t}\mathbf{x_n}$$

(Analogy: $c_1 {a_1}^k \mathbf{x_1} + \cdots + c_n {\lambda_n}^k \mathbf{x_n}$ **solution to difference equations)**

Stability

Not all systems have a steady state

$\Rightarrow$ **eigenvalues of** A **tell us what to expect**

1. **Stability:** $\mathbf{u}(\mathbf{t}) \rightarrow \mathbf{0}$ **when Re(**λ**)** < 0
2. **Steady state: One eigenvalue is** 0 **and all other eigenvalues have negative real parts**
3. **Blow up: Re(**λ**)**> 0 **for any** λ

For 2×2

Fact **For 2×2 matrix** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **system is stable if Re(**λ**)**< 0

$\Leftrightarrow$ **trace** $T = a + d < 0$ $(\lambda_1 + \lambda_2 < 0)$

det $D = ad - bc > 0$ $(\lambda_1 \lambda_2 > 0)$

Reason:

"$\Rightarrow$**"If** λ**'s are real** $\&$ **negative**

sum = $T < 0$, $\lambda_1 \lambda_2 = D > 0$

"$\Leftarrow$" If $D > 0$, $\lambda_1\lambda_2$ has same sign

If $T < 0$, both $\lambda_1, \lambda_2 < 0$

Complex λ's :

$\lambda_1 = r + is$, $\lambda_2 = r - is$ **(otherwise T is not real)**

$D = \lambda_1\lambda_2 = r^2 + s^2 > 0$

$T = \lambda_1 + \lambda_2 = 2r$

so if $T < 0 \Rightarrow$ Re(λ_1), Re(λ_2)< 0

if $r < 0 \Rightarrow T < 0$

Matrix exponential: e^{At}

Q: What does e^{At} mean if A is a matrix ?

Recall: for a real number

$e^x = 1 + x + \frac{x^2}{2!} + \cdots$

Define e^{At} using the same formula

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

Note 1: derivative of e^{At}

$$\frac{de^{At}}{dt} = A + A^2t + \frac{1}{2}A^3t^2 + \cdots = Ae^{At}$$

Note 2: eigenvalues of e^{At}

$$\begin{aligned} e^{At}\mathbf{x} &= (I + At + \frac{(At)^2}{2!} + \cdots)\mathbf{x} \\ &= (1 + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots)\mathbf{x} \\ &= e^{\lambda t}\mathbf{x} \Rightarrow \textbf{eigenvalues} = e^{\lambda t} \end{aligned}$$

Note 3: $e^{At} = Se^{\Lambda t}S^{-1}$

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2!} + \cdots \\ &= SS^{-1} + S\Lambda S^{-1} + S(\frac{\Lambda^2 t^2}{2!})S^{-1} + \cdots \\ &= Se^{\Lambda t}S^{-1} \end{aligned}$$

$$= S \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} S^{-1}$$

(easier way to compute e^{At})

Alternative way to solve: $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$

Note: $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$

(couples the pure solutions)

Let u=Sv (S: matrix of eigenvectors)

$\Rightarrow S\frac{d\mathbf{v}}{dt} = AS\mathbf{v}$

$\Rightarrow \frac{d\mathbf{v}}{dt} = S^{-1}AS\mathbf{v} = \Lambda\mathbf{v}$

This diagonalize the system:

$\frac{dv_i}{dt} = \lambda_i v_i,\ i = 1, \cdots, n$

General solution:

$$\mathbf{v}(t) = e^{\Lambda t}\mathbf{v}(0)$$
$$\Rightarrow S^{-1}\mathbf{u}(t) = e^{\Lambda t}S^{-1}\mathbf{u}(0)$$
$$\Rightarrow \mathbf{u}(t) = Se^{\Lambda t}S^{-1}\mathbf{u}(0) = e^{At}\mathbf{u}(0)$$

Recall:

$$\mathbf{u}(0) = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_n\mathbf{x_n}$$
$$= S\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow S^{-1}\mathbf{u}(0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$
$$\Rightarrow e^{At}\mathbf{u}(0) = Se^{\Lambda t}S^{-1}\mathbf{u}(0)$$
$$= \begin{bmatrix} \mathbf{x_1} & \cdots & \mathbf{x_n} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$
$$= c_1 e^{\lambda_1 t}\mathbf{x_1} + \cdots + c_n e^{\lambda_n t}\mathbf{x_n}$$

(same as before)

(read Ex6, p.321)

Note 1: e^{At} **always has inverse** e^{-At}

Reason: $e^{At} = Se^{\Lambda t}S^{-1}$

$\Rightarrow (e^{At})^{-1} = S(e^{\Lambda t})^{-1}S^{-1}$

$= Se^{-\Lambda t}S^{-1} = e^{-At}$

($-A$ & A have same eigenvectors and eigenvalues with a minus sign)

Note 2: The eigenvalues of e^{At} **are always** $e^{\lambda t}$

Reason: $e^{At} = Se^{\Lambda t}S^{-1}$

$\Rightarrow e^{At}S = Se^{\Lambda t}$

$\Rightarrow$ **eigenvalues** $e^{\lambda_1 t} \cdots e^{\lambda_n t}$

Note 3: When A **is skew-symmetric** e^{At} **is orthogonal** **($A^T = -A$)**

(Inverse = tranpose = e^{-At}**)**

Reason:

$$
\begin{aligned}
e^{At} &= I + At + \tfrac{1}{2!}(At)^2 + \cdots \\
\Rightarrow (e^{At})^T &= I + A^T t + \tfrac{1}{2!}(A^T t)^2 + \cdots \\
&= I + (-A)t + \tfrac{1}{2!}(-At)^2 + \cdots \\
&= e^{-At}
\end{aligned}
$$

(Read Ex5 p.320)

Second order

$y'' + by' + ky = 0$

guess solution $y = e^{\lambda t}$

$\Rightarrow (\lambda^2 + b\lambda + k)e^{\lambda t} = 0$

or we can change it into a 2×2 first-order system

Let $\mathbf{u} = \begin{bmatrix} y' \\ y \end{bmatrix}$

$$\Rightarrow \mathbf{u}' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

$$\Rightarrow \mathbf{u}' = A\mathbf{u}$$

Find eigenvalues of A**:**

$$|A - \lambda I| = \begin{vmatrix} -b-\lambda & -k \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + b\lambda + k = 0$$

(same as before)

eigenvectors: $\mathbf{x_1} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$**,** $\mathbf{x_2} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

$$\Rightarrow \mathbf{u}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

k-th order equation:

we get a k $\times$k matrix:

coeff. of equation in the 1^{st} row $\&$ 1's in the diagonal below that $\&$ the rest of entries = 0

(Read Ex4, p.320)