EEDS 205003 Session 24

Diagonalization & powers of A

We learned eigenvalues & eigenvectors

$\Rightarrow$ We can diagonalize a matrix A using eigenvecotrs if A has n independent eigenvectors

Diagonalizize a matrix: $S^{-1}AS = \Lambda$

$\boxed{Fact}$ Suppose n x n matrix A has n independent eigenvectors $\mathbf{x_1}, \ldots, \mathbf{x_n}$. Put them into columns of an eigenvector matrix S. Then $S^{-1}AS$ is the eigenvalue matrix Λ, i.e.,

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Reason:

$$\begin{aligned} AS &= A\begin{bmatrix} \mathbf{x_1} & \mathbf{x_2} & \dots & \mathbf{x_n} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\mathbf{x_1} & \lambda_2\mathbf{x_2} & \dots & \lambda_n\mathbf{x_n} \end{bmatrix} \\ &= S\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = S\Lambda \end{aligned}$$

Since columns of S are independent

$\Rightarrow$ S is invertible $\Rightarrow$ S^{-1} exists

$AS = S\Lambda \Rightarrow S^{-1}AS = \Lambda$ or $A = S\Lambda S^{-1}$

Note: A can be diagonalize since S has an inverse

$\Rightarrow$ without n independent eigenvectors, we cannot diagonalize

Powers of A

Q: What are the eigenvalue & eigenvectors of A^2?

If $A\mathbf{x} = \lambda\mathbf{x}$
then $A(A\mathbf{x}) = \lambda A\mathbf{x}$
$\Rightarrow A^2\mathbf{x} = \lambda^2\mathbf{x}$
(Eigenvalues of A^2 are squares of eigenvalues of$A$)
(Eigenvectors of A^2 are the same as eigenvectors of A)

Alternatively,
$A = S\Lambda S^{-1}$
$\Rightarrow A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$

Similarly,
$A^k = S\Lambda^k S^{-1}$
(eigenvalues raised to the k^{th} power)

(eigenvectors stay the same)

Note 1: we can multiply eigenvectors by nonzero constants

$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A(c\mathbf{x}) = \lambda(c\mathbf{x})$

$\Rightarrow c\mathbf{x}$ is also an eigenvector

Note 2: there is no connection between invertibility & diagonalizability

- Invertibility: whether eigenvalues $\lambda = 0$ or $\lambda \neq 0$

$\lambda = 0 \Rightarrow A\mathbf{x} = \mathbf{0}$ for some nonzero $\mathbf{x} \Rightarrow$ A is singular

- Diagonalizability: whether we have n independent eigenvectors

A has independent column vector $\Leftrightarrow$ A is invertible

A has independent eigenvectors $\Leftrightarrow$ A is diagonalizable

Note 3: Suppose all eigenvalues $\lambda_1 \dots \lambda_n$ are different

$\Rightarrow$ eigenvectors $\mathbf{x_1} \dots \mathbf{x_n}$ are independent

$\Rightarrow$ A can be diagonalized

Any matrix with no repeated eigenvalues can be diagonalized

Reason: check 2 $\times$ 2 case

Suppose $c_1\mathbf{x_1} + c_2\mathbf{x_2} = \mathbf{0}$ ($\mathbf{x_1}$ & $\mathbf{x_2}$: eigenvector)

multiplied by A $\quad \Rightarrow c_1 A\mathbf{x_1} + c_2 A\mathbf{x_2} = \mathbf{0}$

$$\Rightarrow c_1\lambda_1\mathbf{x_1} + c_2\lambda_2\mathbf{x_2} = \mathbf{0}$$

multiplied by λ_2 $\quad \Rightarrow c_1\lambda_2\mathbf{x_1} + c_2\lambda_2\mathbf{x_2} = \mathbf{0}$

-)

$$c_1(\lambda_1 - \lambda_2)\mathbf{x_1} = \mathbf{0}$$

$$\Rightarrow c_1 = 0 \text{ if } \lambda_1 \neq \lambda_2$$

Similarly, $c_2 = 0$, if $\lambda_1 \neq \lambda_2$. So $\mathbf{x_1}$, $\mathbf{x_2}$ are linear independent

Extend to $n \times n$ matrix

Suppose $c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_n\mathbf{x_n} = \mathbf{0}$

$(\mathbf{x_1},\ \mathbf{x_2},\ \ldots,\ \mathbf{x_n}$: eigenvectors)

multiplied by $(A - \lambda_n)$	$c_1(\lambda_1 - \lambda_n)\mathbf{x_1} + \cdots + c_{n-1}(\lambda_{n-1} - \lambda_n)\mathbf{x_{n-1}} = \mathbf{0}$
multiplied by $(A - \lambda_{n-1})$	$c_1(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n-1})\mathbf{x_1} + \cdots + c_{n-2}(\lambda_{n-2} - \lambda_n)(\lambda_{n-2} - \lambda_{n-1})\mathbf{x_{n-2}} = \mathbf{0}$
$\vdots$	$\vdots$
multiplied by $(A - \lambda_2)$	$c1(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n-1})\cdots(\lambda_1 - \lambda_2)\mathbf{x_1} = \mathbf{0}$

$\Rightarrow\ c_1 = 0$ since $\lambda_i's$ are different

Similarly, $c_2 = c_3 = \cdots = c_n = 0$

$\Rightarrow \mathbf{x_1},\ \mathbf{x_2},\ \cdots,\ \mathbf{x_n}$ are linear independent!

Ex: powers of $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

$det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 0.5$

$(A - \lambda_1 I)\mathbf{x_1} = 0 \Rightarrow \mathbf{x_1} = (0.6, 0.4)$

$(A - \lambda_2 I)\mathbf{x_2} = 0 \Rightarrow \mathbf{x_2} = (1, -1)$

$A = S\Lambda S^{-1}$

$$\Rightarrow \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

same S for A^2

$$\Rightarrow A^2 = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^2 & 0 \\ 0 & 0.5^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

same S for A^k

$$\Rightarrow A^k = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 0.5^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

Limit $k \to \infty$

$$\Rightarrow A^\infty = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

$\boxed{Fact}$

If A has n independent eigenvectors with eigenvalue λ_i, then

$A^k \to 0$ as $k \to \infty$ iff all $|\lambda_i| < 1$

(zero matrix)

Repeated eigenvalues

If A has repeated eigenvalues, it may or may not have independent eigenvectors

Ex1: $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \lambda_1 = \lambda_2 = 1$

$(A - \lambda I)\mathbf{x} = \mathbf{0} \Rightarrow$ any $\mathbf{x}$ would work

$\Rightarrow N(A - I)$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ independent eigenvectors

Ex2: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \lambda_1 = \lambda_2 = 2$

$(A - \lambda I)\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$

$\Rightarrow \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(N(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$ has dim=1)

$\Rightarrow$ only one eigenvector

$\Rightarrow$ no independent eigenvectors

Difference equation $\mathbf{u_{k+1}} = A\mathbf{u_k}$

Starting with $\mathbf{u_0}$

$\mathbf{u_{k+1}} = A\mathbf{u_k}$ is a first-order difference equation

sol: $\mathbf{u_k} = A^k \mathbf{u_0}$

write $\mathbf{u_0}$ as combination of eigenvectors of A
i.e.,

$$\mathbf{u_0} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_n\mathbf{x_n}$$
$$= S\mathbf{c}$$

then

$$A\mathbf{u_0} = c_1\lambda_1\mathbf{x_1} + c_2\lambda_2\mathbf{x_2} + \cdots + c_n\lambda_n\mathbf{x_n}$$
$$= S\Lambda\mathbf{c}$$

and

$$A^k\mathbf{u_0} = c_1{\lambda_1}^k\mathbf{x_1} + c_2{\lambda_2}^k\mathbf{x_2} + \cdots + c_n{\lambda_n}^k\mathbf{x_n}$$
$$= S\Lambda^k\mathbf{c}$$

$$\Rightarrow \mathbf{u_k} = A^k\mathbf{u_0} = c_1{\lambda_1}^k\mathbf{x_1} + \cdots + c_n{\lambda_n}^k\mathbf{x_n} = S\Lambda^k\mathbf{c}$$

Fibonacci sequence

The sequence: $0, 1, 1, 2, 3, 5, 8, 13, \cdots$

$F_{k+2} = F_{k+1} + F_k$ (2^{nd} order difference equation)

Q: How do we solve a 2^{nd} order equation?

convert it into 1^{st}-order equations

Let $\mathbf{u_k} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$, then

$$F_{k+2} = F_{k+1} + F_k$$
$$F_{k+1} = F_{k+1}$$

equivalent to

$$\mathbf{u_{k+1}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u_k}$$

Step1: Find eigenvalues & eigenvectors

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda_1 = \tfrac{1}{2}(1+\sqrt{5}),\ \ \lambda_2 = \tfrac{1}{2}(1-\sqrt{5})$$

since $(A-\lambda I)\mathbf{x} = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\mathbf{x} = \mathbf{0}$

if $\mathbf{x} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \Rightarrow \mathbf{x_1} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix},\ \mathbf{x_2} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

Step2: Find $\mathbf{u_0} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$

$$\mathbf{u_0} = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_1 = -c_2 = \frac{1}{\sqrt{5}}$$

Step3:

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \mathbf{u_k} = c_1{\lambda_1}^k\mathbf{x_1} + c_2{\lambda_2}^k\mathbf{x_2}$$

$$\Rightarrow F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

using eigenvalues & eigenvectors, we obtain closed-form expression

for Fibonacci sequence

Summary: when a sequence evolves overtime following 1^{st} order difference equation $\Rightarrow$ eigenvalues of the system matrix determine long term behavior of the series