EEcS 205003 Session 23

Outline

Ch6 Eigenvalues and Eigenvectors

Eigenvalues & Eigenvectors

Eigenvalues: special numbers associated with a matrix
Eigenvectors: special vectors

Q : How special ?

Almost all vectors change direction when multiplied by A but
Eigenvectors $\boldsymbol{x}$ are in the same direction as $A\boldsymbol{x}$

$\boxed{Def}$ For an eigenvector of A (non-zero)

$A\boldsymbol{x} = \lambda\boldsymbol{x}, \lambda$: eigenvalue
(λ tells whether the special vector $\boldsymbol{x}$ is stretched or shrunk or reversed or left unchanged)

Eigenvalue 0

If eigenvalue $\lambda = 0$, then $\exists$ nonzero $\boldsymbol{x}$ such that $A\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0} \Rightarrow \boldsymbol{x}$ is in nullspace of A
$\Rightarrow$ vectors of eigenvalue 0 makes up $N(A)$
If A is singular, then $\lambda = 0$ is an eigenvalue of A
(otherwise consider null space: $A\boldsymbol{x} = \mathbf{0} = 0\boldsymbol{x} \Rightarrow \boldsymbol{x} = \mathbf{0} \Rightarrow N(A) = \{\mathbf{0}\} \Rightarrow$ contradiction !)

Projection matrix P

Suppose P: projection onto a plane
For any vector on the plane, we have
$P\boldsymbol{x_1} = \boldsymbol{x_1} \Rightarrow \boldsymbol{x_1}$ is an eigenvector with eigenvalue 1
A vector $\boldsymbol{x_2}$ perpendicular to the plane $P\boldsymbol{x_2} = 0 \Rightarrow \boldsymbol{x_2}$ is an eigenvector with eigenvalue 0
(nonzero vector $\boldsymbol{x_2} \in N(A) \Rightarrow A$ singular)
The eigenvectors of P spans the entire space (Not true for any matrix)

Ex: $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

$\lambda = 1 \Rightarrow P\boldsymbol{x} = \boldsymbol{x} \Rightarrow \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 0 \Rightarrow P\boldsymbol{x} = 0 \Rightarrow \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Note: Since $P = P^{\mathsf{T}}$, eigenvectors are perpendicular (will prove this later)

6.1 Introduction to Eigenvalues

Ex: The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 & -1

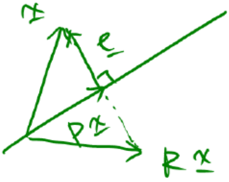
Recall: Eigenvectors for P: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$R\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda = 1$$

$$R\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda = -1$$

$\Rightarrow$ same eigenvectors as P

Why? $R = 2P - I$



$$\underline{e} = (I - P)\underline{x}$$

$$R\underline{x} = \underline{x} - 2\underline{e}$$

$$= \underline{x} - 2(I - P)\underline{x}$$

$$= (2P - I)\underline{x}$$

If $\boldsymbol{x}$ is an eigenvector of P

then $P\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow 2P\boldsymbol{x} = 2\lambda\boldsymbol{x}$

$$-) \quad I\boldsymbol{x} = \boldsymbol{x}$$

$$\begin{aligned} (2P - I)\boldsymbol{x} &= (2\lambda - 1)\boldsymbol{x} \\ \Rightarrow \qquad R\boldsymbol{x} &= (2\lambda - 1)\boldsymbol{x} \end{aligned}$$

So same eigenvector for R but eigenvalue: $\lambda \rightarrow 2\lambda - 1$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} : 2(1) - 1 = 1$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} : 2(0) - 1 = \text{-1}$$

The equation for eigenvalues

An $n \times n$ matrix will have n eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$
Solve $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to obtain eigenvalues & eigenvectors
$\Rightarrow (A - \lambda I)\boldsymbol{x} = \mathbf{0}$
In order for $\boldsymbol{x}$ to be an eigenvector, $A - \lambda I$ must be singular
$\Rightarrow det(A - \lambda I) = 0$ (characteristic polynomial)
(involves only λ, not $\boldsymbol{x}$)

To obtain eigenvectors

For each eigenvalue λ, solve $(A - \lambda I)\boldsymbol{x} = 0$ or $A\boldsymbol{x} = \lambda\boldsymbol{x}$
(in nullspace of $A - \lambda I$)

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (singular)

When A is singular, $\lambda = 0$ is one of eigenvalues
Since $A\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$ has solutions, vectors in $N(A)$ are eigenvectors
By eigenvalue equation,

$$det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - 4$$

$= \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$
$\Rightarrow \lambda = 0$(as expected) or $\lambda = 5$

Now, find eigenvectors

$$(A - 0I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5I)\boldsymbol{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5$$

(Matrix $A - 0I$ & $A - 5I$ are singular since $\lambda = 0, \lambda = 5$ are eigenvalues $(-2, 1), (1, 2)$ are in the nullspaces)

6.1 Introduction to Eigenvalues

Note: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has same eigenvector as $B = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$

$A\boldsymbol{x} = (B + I)\boldsymbol{x} = \lambda\boldsymbol{x} + \boldsymbol{x} = (\lambda + 1)\boldsymbol{x}$

$\Rightarrow$ eigenvalues of A are one plus eigenvalues of B
but eigenvectors stay the same

Bad news:
Elimination does not preserve $\lambda's$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has $\lambda = 0, \lambda = 5$

$U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ has $\lambda = 0, \lambda = 1$

$\boxed{Fact}$ Eigenvalues of U sit on its diagonal (pivots)
Recall: $detU = u_{11} \cdots u_{nn}$
so $det(U - \lambda I) = (u_{11} - \lambda) \cdots (u_{nn} - \lambda) = 0$
$\Rightarrow \lambda = u_{11}, \ \lambda = u_{22}, \cdots, \lambda = u_{nn}$
Eigenvalues are changed during row operations !

Good news: When A is $n \times n$,
(1) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn} = \text{trace}(A)$
For 2×2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc$
$= \lambda^2 - (\text{trace}A)\lambda + detA$

In general, $det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$
from LHS, check coefficient for λ^{n-1}

$$\begin{vmatrix} a_{11} - \lambda & \cdots & \cdots & a_{1n} \\ \vdots & a_{22} - \lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} - \lambda \end{vmatrix}$$

$= (a_{11} - \lambda)C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$
for C_{12}, first row & 2nd column are crossed out $\Rightarrow (a_{11} - \lambda)$, $(a_{22} - \lambda)$ are crossed out $\Rightarrow$ degree at most λ^{n-2}
Similarly for C_{1j}, $j \neq 1$
So λ^{n-1} comes from $(a_{11} - \lambda)\cdots(a_{nn} - \lambda)$
$\Rightarrow$ coefficient for $\lambda^{n-1} = (-1)^n \text{trace} A$
from RHS,
$(\lambda_1 - \lambda)\cdots(\lambda_n - \lambda) \Rightarrow$ coefficient for $\lambda^{n-1} = (-1)^n(\lambda_1 + \cdots + \lambda_n)$
$\Rightarrow \lambda_1 + \cdots + \lambda_n = \text{trace} A$

(2) $det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

$det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda) = 0$

(polynomial of degree n)

Let $\lambda = 0$, we have $det(A) = \lambda_1,\ \lambda_1 \cdots \lambda_n$

A caution:

If $A\boldsymbol{x} = \lambda \boldsymbol{x}, \quad B\boldsymbol{x} = \alpha \boldsymbol{x}$

$\Rightarrow (A + B)\boldsymbol{x} = (\lambda + \alpha)\boldsymbol{x}$

So $A + B$ has eigenvalue $\lambda + \alpha$?

Not really !

Only true when A & B have the same eigenvectors

Similarly, eigenvalues of $AB \neq \lambda(A)\lambda(B)$

Complex eigenvalues

The matrix $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates every vector by 90°

trace $= 0 = \lambda_1 + \lambda_2$, determinant $= 1 = \lambda_1 \lambda_2$

The only real eigenvector is $\mathbf{0}$ since any other vector changes direction when multiplied by Q

$$det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda = i, \text{-}i$$

Note: If $a + bi$ is an eigenvalue $\Rightarrow a - bi$ is also eigenvalue

Note: symmetric matrices have Real eigenvalues

anti-symmetric matrices have Imaginary eigenvalues

($A^\mathsf{T} = -A$, like Q)

Triangular matrix & repeated eigenvalues

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \lambda_1 = 3, \lambda_2 = 3$$

To find eigenvectors,

$$(A - 3I)\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} = \mathbf{0}$$

$\Rightarrow \boldsymbol{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, there is NO independent eigenvector $\boldsymbol{x_2}$