2017 Fall EE203001 Linear Algebra - Midterm 3

1. (16%) Consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

   (a) (8%) Find a Singular Value Decomposition (**SVD**) of **A**

   (b) (5%) Use the result of (a) to find the pseudoinverse of **A**

   (c) (3%) Let $\mathbf{b} = \begin{bmatrix} 0 \\ \sqrt{6} \\ 0 \end{bmatrix}$. Find the least square solution $\hat{\mathbf{x}}$ for $A\hat{\mathbf{x}} = \mathbf{b}$

2. (18%) Let T be a linear transformation on $\mathbb{R}^2$ defined by $T(x) = (x_1 - x_2, 2x_1 + 3x_2)^T$.
   Let $\mathbf{w}_1 = (-1, 1)^T$, $\mathbf{w}_2 = (-2, 1)^T$, $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$.

   (a) (4%) Prove that T is a linear transformation.

   (b) (3%) Find the matrix A representing T with respect to the standard basis **e**.

   (c) (3%) Find the change basis of matrix M from input basis $\boldsymbol{e}$ to output basis $\boldsymbol{w}$.

   (d) (4%) Find the matrix B representing T with respect to $\boldsymbol{w}$.

   (e) (4%) Find the matrix C representing T from input basis $\boldsymbol{e}$ to output basis $\boldsymbol{w}$.

3. (16%) Let matrix $\mathbf{A}=\begin{bmatrix} 4 & 2 \\ 2 & c \end{bmatrix}$.

   (a) (4%) What are the values of c such that **A** is a PD matrix?

   (b) (4%) Given $c = 4$ in A, find the LU decomposition of **A**. Use the LU decomposition to find the sum of squares for $\boldsymbol{x}^T A \boldsymbol{x}$.

   (c) (8%) Please find the axes of the tilted ellipse $4x^2 + 4xy + 4y^2 = 1$. (Hint: Find the sum of squares based on principal axis theorem)

4. (18%) In this problem, we consider similarity between matrices:

   (a) (5%) Is the matrix $A = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$ similar to the matrix $B = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix}$?
   Please find the matrix M such that $B = M^{-1}AM$.

   (b) (3%) If the matrix C is also similar to matrix A with matrix $M = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.
   What are the eigenvectors of matrix C ?

   (c) (6%) If the matrix $E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to the matrix $D = \begin{bmatrix} 6 & 1 \\ -1 & 4 \end{bmatrix}$, find the relationship between a, b, c, d. (Use a and b to express c and d)

   (d) (4%) Use the Jordan form of the matrix D to solve $\frac{d}{dt}\mathbf{u} = J\mathbf{u}$, starting from $\mathbf{u}(0) = (3, 4)$.

5. (14%) An $n \times n$ Walsh matrix W_n is a symmetric matrix consisting of length n orthogonal Walsh codes and is defined recursively as:

$$W_{2n} = \begin{bmatrix} W_n & W_n \\ W_n & \overline{W_n} \end{bmatrix}.$$

Given $W_1 = \begin{bmatrix} 1 \end{bmatrix}$ and $A = \begin{bmatrix} 7.5 & -2.5 & -5 & 1 \\ -2.5 & 7.5 & 1 & -5 \\ -5 & 1 & 7.5 & -2.5 \\ 1 & -5 & -2.5 & 7.5 \end{bmatrix}$:

(a) (6%) Please find W_4 and its 4-point Fourier transform. (Note: $\overline{1} = -1$ and $\overline{-1} = 1$).

(b) (4%) Please decompose A into the form of $R^T R$ (Hint: left multiply A by W_4).

(c) (4%) From (b), we know A is a PD matrix. Is $C = \begin{bmatrix} 16 & -4 & -10 & 2 \\ -4 & 17 & 2 & -9 \\ -10 & 2 & 18 & -5 \\ 2 & -9 & -5 & 19 \end{bmatrix}$ a PD matrix?

Please explain.
(Hint: the sum of PD matrices is also a PD matrix)

6. (18%) Given a real $n \times n$ matrix A

Assume that A is symmetric ($A^T = A$):

(a) (2%) Find the number of negative pivots of AA^T.

(b) (3%) If all the eigenvalues of A are equal to λ, what is the dimension of $N(A - \lambda I)$?

(c) (4%) Find A in (b). (**Hint:** Start from the result in (b))

Assume that A is skew-symmetric ($A^T = -A$):

(d) (2%) Given a complex vector $\mathbf{z}$, find the real part of $\mathbf{z}^H A \mathbf{z}$.

(e) (3%) Show that all the eigenvalues of A are pure imaginary. (**Hint:** Use the result in (d))

(f) (4%) Assume that A is also an orthogonal matrix, its eigenvalues have special properties. Find the eigenvalues and $\det(A)$ for even n.