2017 Fall EE203001 Linear Algebra - Midterm 3 solution

1. (16%) Consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

   (a) (8%) Find a Singular Value Decomposition (**SVD**) of **A**

   (b) (5%) Use the result of (a) to find the pseudoinverse of **A**

   (c) (3%) Let $\mathbf{b} = \begin{bmatrix} 0 \\ \sqrt{6} \\ 0 \end{bmatrix}$. Find the least square solution $\hat{\mathbf{x}}$ for $A\hat{\mathbf{x}} = \mathbf{b}$

Solution:

(a) $A^T A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$det(A^T A - \lambda I) = 0 => \lambda = 3, 1$

$\lambda = 3, \bar{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda = 1, \bar{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$V = [\mathbf{v_1}, \mathbf{v_2}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$AV = U\Sigma$

$\mathbf{u_1} = \mathbf{A v_1 \Sigma} => \mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{3}}} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$

$\mathbf{u_2} = \mathbf{A v_2 \Sigma} => \mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$\mathbf{u_3} \in \mathbf{N(A^T)} = \mathbf{span}\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \}$

$U = \begin{bmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$A = U\Sigma V^T$

(b) $A^+ = V\Sigma^+ U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

(c) $\mathbf{x_0} = \mathbf{A^+ b} => \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{6} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}$

2. (18%) Let T be a linear transformation on $\mathbb{R}^2$ defined by $T(x) = (x_1 - x_2, 2x_1 + 3x_2)^T$. Let $\mathbf{w}_1 = (-1,1)^T$, $\mathbf{w}_2 = (-2,1)^T$, $\mathbf{e}_1 = (1,0)^T$ and $\mathbf{e}_2 = (0,1)^T$.

(a) (4%) Prove that T is a linear transformation.

(b) (3%) Find the matrix A representing T with respect to the standard basis $\mathbf{e}$.

(c) (3%) Find the change basis of matrix M from input basis $\boldsymbol{e}$ to output basis $\boldsymbol{w}$.

(d) (4%) Find the matrix B representing T with respect to $\boldsymbol{w}$.

(e) (4%) Find the matrix C representing T from input basis $\boldsymbol{e}$ to output basis $\boldsymbol{w}$.

Solution:

(a) $\forall \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)\,,\ \mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{R}^2\,,\ \alpha \in \mathbb{R}$

$T(\alpha\mathbf{v} + \mathbf{u}) = T(\alpha\mathbf{v}_1 + \mathbf{u}_1, \alpha\mathbf{v}_2 + \mathbf{u}_2)$

$= (((\alpha\mathbf{v}_1 + \mathbf{u}_1) - (\alpha\mathbf{v}_2 + \mathbf{u}_2)), (2(\alpha\mathbf{v}_1 + \mathbf{u}_1) + 3(\alpha\mathbf{v}_2 + \mathbf{u}_2)))^T$

$= \alpha(\mathbf{v}_1 - \mathbf{v}_2, 2\mathbf{v}_1 + 3\mathbf{v}_2)^T + (\mathbf{u}_1 - \mathbf{u}_2, 2\mathbf{u}_1 + 3\mathbf{u}_2)^T$

$= \alpha T(\mathbf{v}) + T(\mathbf{u})$

$\Rightarrow$ T is a linear

(b) $T(\mathbf{e}_1) = T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = a_1$

$T(\mathbf{e}_2) = T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = a_2$

$\Rightarrow A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

(c) $M = W^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

(d) $B = MAM^{-1} = W^{-1}AW = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 1 & 4 \end{bmatrix}$

(e) $C = MA = W^{-1}A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ -3 & -2 \end{bmatrix}$

3. (16%) Let matrix $\mathbf{A}=\begin{bmatrix} 4 & 2 \\ 2 & c \end{bmatrix}$.

(a) (4%) What are the values of c such that $\mathbf{A}$ is a PD matrix?

(b) (4%) Given $c = 4$ in A, find the LU decomposition of $\mathbf{A}$. Use the LU decomposition to find the sum of squares for $\boldsymbol{x}^T A \boldsymbol{x}$.

(c) (8%) Please find the axes of the tilted ellipse $4x^2 + 4xy + 4y^2 = 1$. (Hint: Find the sum of squares based on principal axis theorem)

Solution:

(a) $\det\mathbf{A} > 0 \Rightarrow 4c - 4 > 0 \Rightarrow c > 1$

(b) $\mathbf{A} = \mathbf{LDL}^T = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \Rightarrow 4x^2 + 4xy + 4y^2 = 4(x + \frac{1}{2}y)^2 + 3y^2$

(c) $4x^2 + 4xy + 4y^2 = 1 \rightarrow \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$

Let $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, $A = Q\Lambda Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

$\rightarrow 4x^2 + 4xy + 4y^2 = 6\left(\frac{x+y}{\sqrt{2}}\right)^2 + 2\left(\frac{-x+y}{\sqrt{2}}\right)^2 = 1$

The axes of the ellipse are $\left(\frac{-1}{2}, \frac{1}{2}\right)$ (or $\left(\frac{1}{2}, \frac{-1}{2}\right)$) and $\left(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)$ (or $\left(\frac{-1}{2\sqrt{3}}, \frac{-1}{2\sqrt{3}}\right)$).

4. (18%) In this problem, we consider similarity between matrices:

(a) (5%) Is the matrix $A = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$ similar to the matrix $B = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix}$?

Please find the matrix M such that $B = M^{-1}AM$.

(b) (3%) If the matrix C is also similar to matrix A with matrix $M = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. What are the eigenvectors of matrix C ?

(c) (6%) If the matrix $E = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to the matrix $D = \begin{bmatrix} 6 & 1 \\ -1 & 4 \end{bmatrix}$, find the relationship between a, b, c, d. (Use a and b to express c and d)

(d) (4%) Use the Jordan form of the matrix D to solve $\frac{d}{dt}\mathbf{u} = J\mathbf{u}$, starting from $\mathbf{u}(0) = (3, 4)$.

Solution:

(a) If A is similar to B, then $\Lambda_A = \Lambda_B$

$\Lambda_A = S_A^{-1} A S_A$, $B = S_B \Lambda_B S_B^{-1} = S_B(S_A^{-1} A S_A) S_B^{-1} = M^{-1} A M$

$$A = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = S_A \Lambda_A S_A^{-1}$$

$$B = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = S_B \Lambda_B S_B^{-1}$$

$$M = S_A S_B^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & -2 \end{bmatrix}$$

(b) $M^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$$M^{-1}\boldsymbol{x}_1 = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$M^{-1}\boldsymbol{x}_2 = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(c) $det(D - \lambda I) = 0 \Rightarrow (6 - \lambda)(4 - \lambda) + 1 = (\lambda - 5)^2 = 0 \Rightarrow \lambda = 5, 5$

$\lambda = 5 \quad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow D$ only has one eigenvector $\Rightarrow D$ can't diagonalized

$\Rightarrow$ use tr(D) and det(D) to find the matrix E.

$tr(D) = 5 + 5 = 10 = a + d \Rightarrow d = 10 - a$

$det(D) = 5 \times 5 = 25 = ad - bc \Rightarrow a(10 - a) - bc = 25$

$\Rightarrow c = \frac{-a^2+10a-25}{b}$

$$\Rightarrow E = \begin{bmatrix} a & b \\ \frac{a^2-10a+25}{b} & 10 - a \end{bmatrix}$$

(d) Jordan form of $D = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

$$\mathbf{u}(0) = \begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \frac{d}{dt}\mathbf{u} = \frac{d}{dt}\begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$$

$$\begin{cases} \frac{dv(t)}{dt} = 5v(t) + w(t) \\ \frac{dw(t)}{dt} = 5w(t) \end{cases}$$

$w(0) = 4 \Rightarrow w(t) = 4e^{5t}$

$v(0) = 3 \Rightarrow v(t) = [v(0) + tw(0)]e^{5t} = (3 + 4t)e^{5t} = 3e^{5t} + 4te^{5t}$

5. (14%) An $n \times n$ Walsh matrix W_n is a symmetric matrix consisting of length n orthogonal Walsh codes and is defined recursively as:

$$W_{2n} = \begin{bmatrix} W_n & W_n \\ W_n & \widetilde{W_n} \end{bmatrix}.$$

Given $W_1 = \begin{bmatrix} 1 \end{bmatrix}$ and $A = \begin{bmatrix} 7.5 & -2.5 & -5 & 1 \\ -2.5 & 7.5 & 1 & -5 \\ -5 & 1 & 7.5 & -2.5 \\ 1 & -5 & -2.5 & 7.5 \end{bmatrix}$:

(a) (6%) Please find W_4 and its 4-point Fourier transform. (Note: $\widetilde{1} = -1$ and $\widetilde{-1} = 1$).

(b) (4%) Please decompose A into the form of $R^T R$ (Hint: left multiply A by W_4).

(c) (4%) From (b), we know A is a PD matrix. Is $C = \begin{bmatrix} 16 & -4 & -10 & 2 \\ -4 & 17 & 2 & -9 \\ -10 & 2 & 18 & -5 \\ 2 & -9 & -5 & 19 \end{bmatrix}$ a PD matrix?

Please explain.
(Hint: the sum of PD matrices is also a PD matrix)

Solution:

(a)

$$W_1 = \begin{bmatrix} 1 \end{bmatrix} \rightarrow W_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$F_4 W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 2+2i & 2-2i \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2-2i & 2+2i \end{bmatrix}$$

(b) Since W_4 is orthogonal and symmatric, $W_4^{-1} = \frac{1}{4} W_4^T$.
Observe from

$$W_4 A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & -4 & 4 & -4 \\ 9 & 9 & -9 & -9 \\ 16 & -16 & -16 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix} W_4,$$

$$A = W_4^{-1} W_4 A = \frac{1}{4} W_4^T W_4 A = \frac{1}{4} W_4^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix} W_4 = \frac{1}{2} W_4^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}^2 \frac{1}{2} W_4 = R^T R.$$

Thus $A = R^T R$, where $R = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 1 & -1 & 1 & -1 \\ 1.5 & 1.5 & -1.5 & -1.5 \\ 2 & -2 & -2 & 2 \end{bmatrix}$.

(c) Known from (b), A is a PD matrix. $e = C - 2A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ can be shown to be a PD matrix by upper-left determinant check. Since the sum of PD matrices is also a PD matrix, $C = e + 2A$ is a PD matrix.

6. (18%) Given a real $n \times n$ matrix A

Assume that A is symmetric ($A^T = A$):

(a) (2%) Find the number of negative pivots of AA^T.

(b) (3%) If all the eigenvalues of A are equal to λ, what is the dimension of $N(A - \lambda I)$?

(c) (4%) Find A in (b). (**Hint:** Start from the result in (b))

Assume that A is skew-symmetric ($A^T = -A$):

(d) (2%) Given a complex vector $\mathbf{z}$, find the real part of $\mathbf{z}^H A\mathbf{z}$.

(e) (3%) Show that all the eigenvalues of A are pure imaginary. (**Hint:** Use the result in (d))

(f) (4%) Assume that A is also an orthogonal matrix, its eigenvalues have special properties. Find the eigenvalues and $\det(A)$ for even n.

Solution:

(a) Let the eigenvalues of A be $\lambda_1, \lambda_2 ... \lambda_n \Rightarrow AA^T = A^2$ has eigenvalues $\lambda_1^2, \lambda_2^2 ... \lambda_n^2 \geq 0$. Since AA^T is also symmetric, # of non-negative pivots = # of non-negative eigenvalues = n, thus # of negative pivots = $n - n = 0$.

(b) Symmetry guarantees that A has n independent eigenvectors, thus dimension of $N(A - \lambda I) = n$.

(c) By dimension theorem, dimension of $C(A - \lambda I) = n -$dimension of $N(A - \lambda I) = n - n = 0 \Rightarrow A - \lambda I$ is a zero matrix $\Rightarrow A = \lambda I$.

(d) Let $\mathbf{x}$ be the real part of $\mathbf{z}$, then $\mathbf{x}^H(A\mathbf{x}) = \mathbf{x}^T(A\mathbf{x}) = (A\mathbf{x})^T\mathbf{x} = \mathbf{x}^T A^T \mathbf{x} = -\mathbf{x}^T A\mathbf{x} = 0$. So the real part of $\mathbf{z}^H A\mathbf{z}$ is zero for any complex vector $\mathbf{z}$.

(e) Let $\mathbf{z}$ be any complex vector, $A\mathbf{z} = \lambda\mathbf{z}, \mathbf{z}^H A\mathbf{z} = \mathbf{z}^H \lambda \mathbf{z} = \lambda \mathbf{z}^H \mathbf{z} = \lambda \|\mathbf{z}\|^2$ has no real part, thus all eigenvalues of A are purely imaginary.

(f) Since A is skew-symmetric and orthogonal, we can show that: **1.** Using the result of (e), all eigenvalues of A are pure imaginary. **2.** $\|A\mathbf{z}\| = \|\mathbf{z}\| = \|\lambda\mathbf{z}\| \Rightarrow |\lambda| = 1$. By the two facts, λ should be either i or $-i$. So when n is even, $\det(A)$=$i(-i)i(-i)... = 1$.