EECS205000: Linear Algebra **Spring 2019**
College of Electrical Engineering and Computer Science
National Tsing Hua University

Homework #3 Solutions
Coverage: Chapter 1–6

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Problem 1. (20 points)

(a) (10 points) Let $\mathbf{P}$ and $\mathbf{Z}$ be positive semidefinite matrices such that $\mathbf{P}^2 = \mathbf{Z}^2$. Then $\mathbf{P} = \mathbf{Z}$.

(b) (10 points) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be invertible. Using the result in part (a), prove that there exist unique matrices $\mathbf{Q}$ and $\mathbf{P}$ which are orthogonal and positive-semidefinite matrices, respectively such that $\mathbf{A} = \mathbf{QP}$.

Solution:

(a) Let $\alpha = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an orthonormal basis consisting of eigenvectors of $\mathbf{P}$. then we have

$$\mathbf{P}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad \forall\, \mathbf{x}_i \in \alpha.$$

Then we can write

$$\mathbf{P}^2\mathbf{x}_i = \lambda_i^2 \mathbf{x}_i = \mathbf{Z}^2\mathbf{x}_i, \quad \forall\, \mathbf{x}_i \in \alpha. \tag{1}$$

Consider the case $\lambda_i = 0$. We have

$$\|\mathbf{Z}\mathbf{x}_i\|^2 = \mathbf{x}_i^*\mathbf{Z}^T\mathbf{Z}\mathbf{x}_i = \mathbf{x}_i^*\mathbf{Z}^2\mathbf{x}_i = \mathbf{x}_i^*\mathbf{P}^2\mathbf{x}_i = \|\mathbf{P}\mathbf{x}_i\|^2 = 0,$$

which implies that $\mathbf{Z}\mathbf{x}_i = \mathbf{P}\mathbf{x}_i = \mathbf{0}$. Thus $\mathbf{Z}$ and $\mathbf{P}$ have the identical eigenspaces.
Now, consider the case $\lambda_i > 0$. By (1) we have

$$(\mathbf{Z}^2 - \lambda_i^2\mathbf{I})\mathbf{x}_i = (\mathbf{Z} + \lambda_i\mathbf{I})(\mathbf{Z} - \lambda_i\mathbf{I})\mathbf{x}_i = \mathbf{0}. \tag{2}$$

Here, $\mathbf{Z}+\lambda_i\mathbf{I}$ is invertible since $det(\mathbf{Z}+\lambda_i\mathbf{I}) \neq 0$ (otherwise, $-\lambda_i$ is a eigenvalue of $\mathbf{Z}$ which is impossible). Then we have

$$(\mathbf{Z} - \lambda_i\mathbf{I})\mathbf{x}_i = \mathbf{0},$$

which implies $\mathbf{P}\mathbf{x}_i = \lambda_i\mathbf{x}_i = \mathbf{Z}\mathbf{x}_i$. This means that $\mathbf{P}$ and $\mathbf{Z}$ meet on the same basis α and therefore $\mathbf{P} = \mathbf{Z}$. ∎

(b) By singular value decomposition we have $\mathbf{A} = \mathbf{U\Sigma V}^T$ where $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{V}^T\mathbf{V} = \mathbf{I}$. Then we can write

$$\mathbf{A} = (\mathbf{U}\mathbf{V}^T)(\mathbf{V\Sigma V}^T) = \mathbf{QP},$$

where $\mathbf{Q} \triangleq \mathbf{U}\mathbf{V}^T$ is an orthogonal matrix and $\mathbf{P} \triangleq \mathbf{V\Sigma V}^T$ is a positive-semidefinite matrix. By contradiction assume $\mathbf{A} = \mathbf{QP} = \mathbf{UZ}$ where $\mathbf{Q}$, $\mathbf{U}$ are orthogonal matrices and $\mathbf{P}$, $\mathbf{Z}$ are positive-semidefinite matrices. Since $\mathbf{A}$ is invertible, $\mathbf{P}$ and $\mathbf{Z}$ are also invertible. Then we have

$$\mathbf{U}^T\mathbf{Q} = \mathbf{U}^T\mathbf{QPP}^{-1} = \mathbf{U}^T\mathbf{UZP}^{-1} = \mathbf{ZP}^{-1}.$$

Since $\mathbf{Q}$, $\mathbf{U}$ are orthogonal matrices, then $\mathbf{ZP}^{-1}$ is also an orthogonal matrix. Then we have

$$(\mathbf{ZP}^{-1})^T\mathbf{ZP}^{-1} = \mathbf{P}^{-1}\mathbf{Z}^T\mathbf{ZP}^{-1} = \mathbf{I},$$

which implies that $\mathbf{Z}^2 = \mathbf{P}^2$. Since $\mathbf{Z}$ and $\mathbf{P}$ are positive-semidefinite matrices, then by part (a) we have $\mathbf{Z} = \mathbf{P}$. ∎

Problem 2. (15 points) Let $\mathbf{A}$ be an invertible matrix.

(a) (5 points) Show that for any eigenvalue λ of $\mathbf{A}$, λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.

(b) (5 points) Show that the eigenspace of $\mathbf{A}$ corresponding to λ is the same as the eigenspace of $\mathbf{A}^{-1}$ corresponding to λ^{-1}.

(c) (5 points) Show that if $\mathbf{A}$ is invertible and diagonalizable, then $\mathbf{A}^{-1}$ is also diagonalizable.

Solution:

(a) We know $\mathbf{Ax} = \lambda\mathbf{x}$ where $\lambda \neq 0$ (otherwise if $\lambda = 0$, $\mathbf{Ax} = 0$, $\mathbf{x} \neq \mathbf{0}$ which implies $\mathbf{A}$ is NOT INVERTIBLE). Then we can write

$$\mathbf{A}^{-1}\mathbf{Ax} = \lambda\mathbf{A}^{-1}\mathbf{x} \implies \lambda^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}.$$

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(b) Let E_λ be the eigenspace of $\mathbf{A}$ corresponding to λ and $E_{\lambda^{-1}}$ be the eigenspace of $\mathbf{A}^{-1}$ corresponding to λ^{-1}. If $\mathbf{x} \in E_\lambda$, we have

$$\mathbf{Ax} = \lambda\mathbf{x} \implies \lambda^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x},$$

and hence $\mathbf{x} \in E_{\lambda^{-1}}$. Conversely, if $\mathbf{x} \in E_{\lambda^{-1}}$ we have

$$\mathbf{A}^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x} \implies \mathbf{x} = \lambda^{-1}\mathbf{Ax} \implies \mathbf{Ax} = \lambda\mathbf{x},$$

and hence $\mathbf{x} \in E_\lambda$. ■

(c) Let $\mathbf{A}$ be diagonalizable and we can write $\mathbf{A} = \mathbf{SDS}^{-1}$ where $\mathbf{D} = \text{Diag}\left([\lambda_1, \ldots, \lambda_n]\right)$ is a diagonal matrix containing the eigenvalues of $\mathbf{A}$. Since $\mathbf{A}$ and $\mathbf{S}$ are invertible, then we have

$$\mathbf{A}^{-1} = (\mathbf{SDS}^{-1})^{-1} = \mathbf{SD}^{-1}\mathbf{S}^{-1},$$

where $\mathbf{D}^{-1} = \text{Diag}\left([\lambda_1^{-1}, \ldots, \lambda_n^{-1}]\right)$. Thereby, $\mathbf{A}^{-1}$ is diagonalizable. ■

Problem 3. (20 points) The matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ can be written as $\mathbf{A} = \mathbf{LDU}$, where $\mathbf{L}$ and $\mathbf{U}$ are lower unitriangular and upper unitriangular matrices, respectively, and $\mathbf{D}$ is a diagonal matrix. Show that LDU decomposition can be reduced to $\mathbf{A} = \mathbf{LDL}^T$ if $\mathbf{A}$ is a nonsingular symmetric matrix.

Solution:

Suppose $\mathbf{A}$ is a nonsingular symmetric matrix. By LDU decomposition we have $\mathbf{A} = \mathbf{LDU}$ where $\mathbf{L}$ and $\mathbf{U}$ are lower unitriangular and upper unitriangular matrices (and hence invertible) with all the diagonal entries equal to unity, respectively, and $\mathbf{D}$ is a diagonal matrix. Clearly, $(\mathbf{U}^{-1})^T\mathbf{AU}^{-1}$ is symmetric and we can write

$$(\mathbf{U}^{-1})^T\mathbf{AU}^{-1} = (\mathbf{U}^{-1})^T\mathbf{LDUU}^{-1} = (\mathbf{U}^{-1})^T\mathbf{LD}.$$

Since $(\mathbf{U}^{-1})^T\mathbf{LD}$ is symmetric and $(\mathbf{U}^{-1})^T$ is lower unitriangular matrix, those imply that $(\mathbf{U}^{-1})^T\mathbf{LD}$ is a diagonal matrix. Besides, $(\mathbf{U}^{-1})^T\mathbf{L}$ is a diagonal matrix since $\mathbf{D}$ is diagonal matrix. Due to that $(\mathbf{U}^{-1})^T$ and $\mathbf{L}$ are lower unitriangular matrices, it implies that $(\mathbf{U}^T)^{-1}\mathbf{L} = \mathbf{I}_n$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Hence, $\mathbf{L} = \mathbf{U}^T$. ■

Problem 4. (15 points) The Frobenius norm defined for $\mathbf{A} \in \mathbb{C}^{n\times n}$ by $\|\mathbf{A}\|_F = \left(\text{Tr}(\mathbf{A}^H\mathbf{A})\right)^{1/2}$ where $\text{Tr}(\cdot)$ denotes the trace of a matrix. Show that

$$\|\mathbf{A}\|_F \leq (\text{rank}(\mathbf{A}))^{1/2}\|\mathbf{A}\|_2,$$

where $\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{Ax}\|_2$ and $\mathbf{x}$ is an $n \times 1$ vector.

Solution:

We know that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^H\mathbf{A})$ and the eigenvalues of $\mathbf{A}^H\mathbf{A}$ are ordered as $\lambda_{max}(\mathbf{A}^H\mathbf{A}) = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0$. Then we have

$$\|\mathbf{A}\|_F = (\text{Tr}(\mathbf{A}^H\mathbf{A}))^{1/2} = \left(\sum_{i=1}^{n} \lambda_i(\mathbf{A}^H\mathbf{A})\right)^{1/2} \leq (\text{rank}(\mathbf{A}))^{1/2}\lambda_{max}^{1/2}(\mathbf{A}^H\mathbf{A}).$$

On the other hand, since $\mathbf{A}^H\mathbf{A}$ is Hermitian we can write $\mathbf{A}^H\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^H$ where $\mathbf{U}$ is a unitary matrix and $\mathbf{D}$ is diagonal. Then we have

$$\|\mathbf{A}\|_2^2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2^2 = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^H\mathbf{A}^H\mathbf{A}\mathbf{x} = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^H\mathbf{U}\mathbf{D}\mathbf{U}^H\mathbf{x} = \max_{\|\mathbf{y}\|_2=1} \mathbf{y}^H\mathbf{D}\mathbf{y} = \max_{\|\mathbf{y}\|_2=1} \sum_{i=1}^{n} \lambda_i|y_i|^2$$
$$\leq \lambda_{max}(\mathbf{A}^H\mathbf{A}),$$

and for $\mathbf{x}$ to be the corresponding eigenvector of $\lambda_{max}(\mathbf{A}^H\mathbf{A})$, $\|\mathbf{A}\mathbf{x}\|_2$ attains its maximum and hence $\|\mathbf{A}\|_2 = \lambda_{max}^{1/2}(\mathbf{A}^H\mathbf{A})$ and thereby

$$\|\mathbf{A}\|_F \leq (\text{rank}(\mathbf{A}))^{1/2}\|\mathbf{A}\|_2.$$

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Problem 5. (5 points) Let $\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 3 & 2 \end{bmatrix}$. Find a general formula based on n (a positive integer) for $\mathbf{A}^n$.

Solution:

The idea to solve this problem is by matrix diagonalization where $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$. For that the characteristic polynomial is

$$det\,(\mathbf{A} - \lambda\mathbf{I}) = det\left(\begin{bmatrix} 6-\lambda & -1 \\ 3 & 2-\lambda \end{bmatrix}\right) = \lambda^2 - 8\lambda + 15 = 0, \tag{3}$$

which implies $\lambda_1 = 5$ and $\lambda_2 = 3$. Hence, the eigenvalue matrix is

$$\mathbf{D} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}. \tag{4}$$

Then we can obtain the matrix containing the eigenvectors of $\mathbf{A}$ as

$$\mathbf{U} = \begin{bmatrix} 1 & 1/3 \\ 1 & 1 \end{bmatrix}. \tag{5}$$

And its inverse can be obtained as

$$\mathbf{U}^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -3 & 3 \end{bmatrix}. \tag{6}$$

Then we can write

$$\mathbf{A}^n = (\mathbf{U}\mathbf{D}\mathbf{U}^{-1})^n = (\mathbf{U}\mathbf{D}\mathbf{U}^{-1})(\mathbf{U}\mathbf{D}\mathbf{U}^{-1})\ldots(\mathbf{U}\mathbf{D}\mathbf{U}^{-1}) = \mathbf{U}\mathbf{D}^n\mathbf{U}^{-1}, \tag{7}$$

and hence we have

$$\mathbf{A}^n = \mathbf{U}\begin{bmatrix} 5^n & 0 \\ 0 & 3^n \end{bmatrix}\mathbf{U}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1/3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -3 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3\times 5^n - 3^n & 5^n + 3^n \\ 3\times 5^n - 3^{n+1} & 5^n + 3^{n+1} \end{bmatrix}. \tag{8}$$

Problem 6. (5 points) Define the matrix $\mathbf{A}$ as

$$n \times 2n \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 9 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 9 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{0 \quad 0 \quad \cdots \quad 9}_{2n \times n} & & & & 0 & 0 & \cdots & 1 \end{bmatrix} \right.$$

be a $2n \times 2n$ matrix. Find $\mathbf{A}^\ell$ where ℓ is a positive integer number.

Solution:

This matrix can be easily written in block matrix form as

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ 9\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}.$$

Then we have

$$\mathbf{A}^2 = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ 9\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ 9\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ 18\mathbf{I}_n & \mathbf{I}_n \end{bmatrix},$$

$$\mathbf{A}^3 = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ 18\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ 9\mathbf{I}_n & \mathbf{I}_n \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ 27\mathbf{I}_n & \mathbf{I}_n \end{bmatrix},$$

and finally

$$\mathbf{A}^\ell = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_n \\ 9\ell\mathbf{I}_n & \mathbf{I}_n \end{bmatrix}.$$

Problem 7. (5 points) Let V be a finite dimensional vector space with $dim(V) = n$. Prove that every basis for V contains the same number of vectors.

Solution:

Suppose α is a basis for V containing n vectors and β be any arbitrary basis for V containing m vectors. Then we have:
case 1: if $m > n$; then, there is a subset $\mathcal{S} \subset \beta$ containing $n+1$ vectors. Since $\mathcal{S}$ is linearly independent and $V = span\{\alpha\}$, then $n+1 \leq n$ which is a clear contradiction. Hence $m \leq n$.
case 2: if $m < n$; then, there is a subset $\mathcal{S} \subset \alpha$ containing $m+1$ vectors. Since $\mathcal{S}$ is linearly independent and $V = span\{\beta\}$, then $m+1 \leq m$ which is a clear contradiction. Hence $m \geq n$. Then by case 1 and case 2 we have $n = m$ and the proof is completed. ■

Problem 8. (10 points) Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Find $\mathbf{J}$ and invertible $\mathbf{S}$ such that $\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ where $\mathbf{J}$ is the Jordan canonical form of $\mathbf{A}$.

Solution:

First, we need to find the eigenvalues and then we have

$$(2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 2 & 4-\lambda \end{vmatrix} = 0,$$

after simplification we have $\lambda_1 = 2$, $\lambda_2 = 2$ and $\lambda_3 = 5$. The we have

$$N(\mathbf{A} - 2\mathbf{I}) = span\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Besides,

$$N(\mathbf{A} - 5\mathbf{I}) = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Then the Jordan matrix is

$$\mathbf{J} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Moreover, let $\mathbf{S} = [\mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{s}_3]$ and for this Jordan decomposition we can write

$$\mathbf{AS} = \mathbf{SJ},$$

which implies that

$$[\mathbf{As}_1 \quad \mathbf{As}_2 \quad \mathbf{As}_3] = [2\mathbf{s}_1 \quad \mathbf{s}_1 + 2\mathbf{s}_2 \quad 5\mathbf{s}_3],$$

and hence

$$\mathbf{As}_1 = 2\mathbf{s}_1 \implies (\mathbf{A} - 2\mathbf{I})\mathbf{s}_1 = \mathbf{0}, \tag{9}$$

$$\mathbf{As}_2 = \mathbf{s}_1 + 2\mathbf{s}_2 \implies (\mathbf{A} - 2\mathbf{I})\mathbf{s}_2 = \mathbf{s}_1, \tag{10}$$

$$\mathbf{As}_3 = 5\mathbf{s}_3 \implies (\mathbf{A} - 5\mathbf{I})\mathbf{s}_3 = \mathbf{0}, \tag{11}$$

$$\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3. \text{ are linearly independent.} \tag{12}$$

From (9), (10) implies that $(\mathbf{A} - 2\mathbf{I})^2\mathbf{s}_2 = \mathbf{0}$ and together with (2) implies that $\mathbf{s}_2 \in N\left((\mathbf{A} - 2\mathbf{I})^2\right)$ but $\mathbf{s}_2 \notin N(\mathbf{A} - 2\mathbf{I})$, thereby we have

$$\mathbf{s}_2 = \begin{bmatrix} 1/3 \\ -1/3 \\ 0 \end{bmatrix}.$$

Then

$$\mathbf{s}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \in N(\mathbf{A} - 2\mathbf{I}).$$

Then choose $\mathbf{s}_3$ such that $\mathbf{s}_3 \in N(\mathbf{A} - 5\mathbf{I})$ and being linearly independent with $\mathbf{s}_1$ and $\mathbf{s}_2$ as

$$\mathbf{s}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence the matrices can be obtained as

$$\mathbf{S} = \begin{bmatrix} -1 & 1/3 & 1 \\ 0 & -1/3 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{S}^{-1} = \begin{bmatrix} -1/3 & -1/3 & 2/3 \\ 1 & -2 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Problem 9. (10 points) Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

(a) (5 points) Find the complete solution for the equation $\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t)$ with the initial condition $\mathbf{x}(0) = [0\ \ 2]^T$.

(b) (5 points) Find the singular value decomposition as $\mathbf{A} = \mathbf{U\Sigma V}^T$ using diagonalization of the matrix $\mathbf{A}^T\mathbf{A}$.

Solution:

(a) We need first to find the eigenvalues and eigenvector of this matrix. We have $\mathbf{A} = \mathbf{Q\Lambda Q}^{-1}$ where

$$\mathbf{\Lambda} = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}, \quad \mathbf{Q}^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}.$$

We know the complete solution for the above linear differential equation is as follows

$$\mathbf{x}(t) = C_1 e^{(2+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + C_2 e^{(2-i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix},$$

where by the initial condition we can obtain $C_1 = C_2 = 1$. Hence, we have

$$\mathbf{x}(t) = e^{(2+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} + e^{(2-i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

(b) To find the singular value decomposition, we have

$$\mathbf{A}^T\mathbf{A} = \mathbf{V\Sigma}^T\mathbf{U}\mathbf{U}^T\mathbf{\Sigma V}^T = \mathbf{V\Sigma}^2\mathbf{V}^T = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathbf{V} = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. Clearly, we have $\mathbf{A}\mathbf{u}_1 = \sigma_1\mathbf{v}_1$ and $\mathbf{A}\mathbf{u}_2 = \sigma_2\mathbf{v}_2$ where $\sigma_1 = \sigma_2 = \sqrt{5}$. Then we can obtain

$$\mathbf{u}_1 = \frac{\sqrt{5}}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \frac{\sqrt{5}}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

And therefore,

$$\mathbf{U} = \frac{\sqrt{5}}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$