EECS205000: Linear Algebra **Spring 2019**
College of Electrical Engineering and Computer Science
National Tsing Hua University

Homework #1 Solutions
Coverage: Chapter 1–3

Instructor: Chong-Yung Chi *TAs: Amin Jalili, Yi-Wei Li & Ping-Rui Chiang*

Problem 1. (35 points) Which of the following statements is TRUE and which one is FALSE? Justify your answers.

(i) Let $V \triangleq \{\mathbf{x} = (x_1, x_2, x_3) \mid \mathbf{x} \in \mathbb{R}^3\}$. Let $\mathbf{y} = (y_1, y_2, y_3), \mathbf{w} = (w_1, w_2, w_3) \in V$, $t \in \mathbb{R}$ and consider

$$\begin{aligned} \mathbf{y} + \mathbf{w} &\triangleq (y_1 + w_1, y_2 + 2w_2, y_3 - 3w_3), \\ t\mathbf{y} &\triangleq (ty_1, ty_2, ty_3). \end{aligned}$$

Then V is a subspace.

(ii) The set $\mathcal{A}$ of all 2×2 lower triangular matrices forms a subspace for the space $\mathbb{R}^{2\times 2}$.

(iii) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. Suppose the system $\mathbf{A}\mathbf{x} = \mathbf{0}_n$ has infinitely many solutions and $\mathbf{B}\mathbf{x} = \mathbf{0}_n$ has one solution. This implies the system $\mathbf{A}\mathbf{B}\mathbf{x} = \mathbf{0}_n$ has exactly one solution.

(iv) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. If $\mathbf{x}$ is in the nullspace of $\mathbf{A}$, then $\mathbf{x}$ is in the nullspace of $\mathbf{A}^2$.

(v) The set $\mathcal{B} \triangleq \{(a, b, c) \in \mathbb{R}^3 \mid a = 4b\}$ is not a subspace of $\mathbb{R}^3$.

(vi) Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 6 \end{bmatrix}$. Suppose $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. This linear system of equations is not consistent for any $\mathbf{b}$ (i.e., no solution for any $\mathbf{b}$).

(vii) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{A}\mathbf{B} = \mathbf{I}_n$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Then, $rank(\mathbf{A}) = n$.

Solution:

(i) FALSE. Let $\mathbf{y} = (y_1, y_2, y_3), \mathbf{w} = (w_1, w_2, w_3), \mathbf{z} = (z_1, z_2, z_3) \in V$, then we have

$$\begin{aligned} (\mathbf{y} + \mathbf{w}) + \mathbf{z} &= (y_1 + w_1 + z_1, y_2 + 2w_2 + 2z_2, y_3 - 3w_3 - 3z_3), \\ \mathbf{y} + (\mathbf{w} + \mathbf{z}) &= (y_1 + w_1 + z_1, y_2 + 2w_2 + 4z_2, y_3 - 3w_3 + 9z_3), \\ \Longrightarrow\ & (\mathbf{y} + \mathbf{w}) + \mathbf{z} \neq \mathbf{y} + (\mathbf{w} + \mathbf{z}), \end{aligned}$$

and hence V lacks the associativity of addition and is not a vector space.

(ii) TRUE. Clearly $\mathbf{0}_{2\times 2} \in \mathcal{A}$. Then, Let $\mathbf{A}_1 = \begin{bmatrix} a_1 & 0 \\ c_1 & d_1 \end{bmatrix} \in \mathcal{A}$ and $\mathbf{A}_2 = \begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix} \in \mathcal{A}$. Then, we have

$$\mathbf{A}_1 + \mathbf{A}_2 = \begin{bmatrix} a_1 & 0 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 0 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \in \mathcal{A}.$$

Also,

$$\alpha\mathbf{A}_1 = \begin{bmatrix} \alpha a_1 & 0 \\ \alpha c_1 & \alpha d_1 \end{bmatrix} \in \mathcal{A}.$$

where $\alpha \in \mathbb{R}$. Therefore, $\mathcal{A}$ is a subspace of $\mathbb{R}^{2\times 2}$.

(iii) FALSE. Since the system $\mathbf{Ax} = \mathbf{0}_n$ has infinitely many solutions, this implies $\mathbf{A}$ has a nontrivial nullspace. Moreover, the system $\mathbf{Bx} = \mathbf{0}_n$ has one solution and this implies $\mathbf{B}$ is invertible. Then, for all $\mathbf{x}$ such that $\mathbf{Bx} \in N(\mathbf{A})$ (i.e., $\mathbf{Bx}$ is in the null space of $\mathbf{A}$), we have $\mathbf{ABx} = \mathbf{0}_n$. Hence, the system $\mathbf{ABx} = \mathbf{0}_n$ has infinitely many solutions.

(iv) True. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. If $\mathbf{x}$ is in the nullspace of $\mathbf{A}$ (i.e., $\mathbf{x} \in N(\mathbf{A})$) we have

$$\mathbf{Ax} = \mathbf{0}_n,$$

and then by multiplying from the left side with $\mathbf{A}$ we have

$$\mathbf{A}^2\mathbf{x} = \mathbf{A0}_n = \mathbf{0}_n,$$

and clearly $\mathbf{x} \in N(\mathbf{A}^2)$.

(v) FALSE. Clearly $\mathbf{0}_3 \in \mathcal{B}$. Let $\mathbf{u}_1 = (4b_1, b_1, c_1)$ and $\mathbf{u}_2 = (4b_2, b_2, c_2) \in \mathcal{B}$. Then, $\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = (4(\alpha b_1 + \beta b_2), \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2) \in \mathcal{B}$ and hence it is a subspace.

(vi) FALSE. By reduced row echelon form we have

$$\left[\begin{array}{cc|c} 1 & 3 & b_1 \\ 2 & 4 & b_2 \\ 2 & 6 & b_3 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cc|c} 1 & 0 & -2b_1 + 3b_2/2 \\ 0 & 1 & b_1 - b_2/2 \\ 0 & 0 & b_3 - 2b_1 \end{array}\right],$$

which implies that this system is consistent when $b_3 = 2b_1$.

(vii) TRUE.

Claim 1: $rank(\mathbf{AB}) \leq rank(\mathbf{A})$.
Proof: Suppose by contradiction $rank(\mathbf{AB}) > rank(\mathbf{A})$. Let $C(\mathbf{A}) = \{\mathbf{Ax} \mid \mathbf{x} \in \mathbb{R}^{n\times n}\}$ and $C(\mathbf{AB}) = \{\mathbf{ABx} \mid \mathbf{x} \in \mathbb{R}^{n\times n}\}$ be the column spaces of $\mathbf{A}$ and $\mathbf{AB}$, respectively. Then by assumption there exists $\mathbf{y} \in C(\mathbf{AB})$ but $\mathbf{y} \notin C(\mathbf{A})$. Then, we have

$$\mathbf{ABz} = \mathbf{y},$$

where $\mathbf{z} \in \mathbb{R}^n$. Clearly, $\mathbf{Aw} = \mathbf{y}$ where $\mathbf{w} \triangleq \mathbf{Bz}$ and hence $\mathbf{y} \in C(\mathbf{A})$. This is a clear contradiction to the first assumption and thereby $rank(\mathbf{AB}) \leq rank(\mathbf{A})$.

Clearly by Claim 1 we have $n = rank(\mathbf{I}_n) = rank(\mathbf{AB}) \leq rank(\mathbf{A})$. On the other hand, for $\mathbf{A} \in \mathbb{R}^{n\times n}$ we have $rank(\mathbf{A}) \leq n$. Hence, $rank(\mathbf{A}) = n$.

□

Problem 2. (5 points) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{bmatrix}$. Use elimination steps to find the matrix $\mathbf{E}$ such that $\mathbf{EA} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Solution:

By considering $\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$ we have

$$\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 2 & 4 & -2 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then by $\mathbf{E}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ we have

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

By considering $\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ we obtain

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

By $\mathbf{E}_4 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ we have

$$\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Finally, by $\mathbf{E}_5 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ we have

$$\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, $\mathbf{EA} = \mathbf{I}$ where $\mathbf{E} = \mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 = \begin{bmatrix} 4 & 0 & -1 & -1 \\ 8 & -1 & -3 & -2 \\ -4 & 1/2 & 1 & 1 \\ 1 & -1/2 & -1 & 0 \end{bmatrix}$.

Problem 3. (10 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 7 & 8 & 4 \end{bmatrix}$.

(i) (5 points) Find a decomposition of $\mathbf{A}$ such that $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ is a lower unitriangular matrix and $\mathbf{U}$ is an upper triangular matrix.

(ii) (5 points) Consider the system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 2 & 6 & 19 \end{bmatrix}^T$. Solve this system using the resulting LU decomposition in part (i).

Solution:

(i) By considering $\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$ we have

$$\mathbf{M}_1\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -6 & -17 \end{bmatrix}.$$

Then, by $\mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ we have

$$\mathbf{M}_2\mathbf{M}_1\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & 0 & -9 \end{bmatrix}.$$

Clearly, $\mathbf{U} \triangleq \mathbf{M}_2\mathbf{M}_1\mathbf{A}$. And then,

$$\mathbf{A} = \mathbf{M}_1^{-1}\mathbf{M}_2^{-1}\mathbf{U},$$

which implies $\mathbf{A} = \mathbf{LU}$ where

$$\mathbf{L} \triangleq \mathbf{M}_1^{-1}\mathbf{M}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}.$$

(ii) Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$. Then, we have

$$\mathbf{Ax} = \mathbf{LUx} = \mathbf{b}.$$

Suppose $\mathbf{y} \triangleq \mathbf{Ux}$. Then,

$$\mathbf{Ly} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 19 \end{bmatrix},$$

Using forward substitution, we can find

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 9 \end{bmatrix}.$$

Finally, doing backward substitution we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Problem 4. (15 points) Let V be a finite dimensional vector space and $W_1, W_2 \subset V$ be two subspaces with dimensions p and q, respectively.

(i) (5 points) Prove that $W_1 \cap W_2$ is the *largest subspace* of V contained in both W_1 and W_2.

(ii) (5 points) Assume $p \geq q$. Show that:

$$dim(W_1 \cap W_2) \leq q.$$

(iii) (5 points) Let $V = \mathbb{R}^3$ and assume $p > q > 0$. Let $W_1 + W_2 \triangleq \{\mathbf{w}_1 + \mathbf{w}_2 \mid \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}$. Find an example of subspaces W_1 and W_2 such that

$$dim(W_1 + W_2) = p + q.$$

Solution:

(i) Let's first show that $W_1 \cap W_2$ is a subspace. Clearly, $\mathbf{0} \in W_1 \cap W_2$. Let $\mathbf{u}, \mathbf{v} \in W_1 \cap W_2$. Since W_1 is a subspace of V and $W_1 \cap W_2 \subset W_1$, then $\mathbf{u} + \mathbf{v} \in W_1$ and $c\mathbf{u} \in W_1$ where $c \in \mathbb{R}$. Similarly, $\mathbf{u} + \mathbf{v} \in W_2$ and $c\mathbf{u} \in W_2$. This implies $\mathbf{u} + \mathbf{v} \in W_1 \cap W_2$ and $c\mathbf{u} \in W_1 \cap W_2$. Hence, $W_1 \cap W_2$ is closed under addition and scalar multiplication and then $W_1 \cap W_2$ is a subspace of V. Now, we need to show that for any arbitrary subspace $U \subset V$ such that $U \subseteq W_1$ and $U \subseteq W_2$, then $U \subseteq W_1 \cap W_2$. It is true since $\mathbf{u} \in W_1$ and $\mathbf{u} \in W_2$ implies $\mathbf{u} \in W_1 \cap W_2$. Therefore, $W_1 \cap W_2$ is the largest subspace of V contained in W_1 and W_2 and the proof is completed. □

(ii) Clearly, $W_1 \cap W_2 \subset W_2$ and hence $dim(W_1 \cap W_2) \leq dim(W_2) = q$.

□

(iii) Suppose W_1 be the xy-plane with dimension $p = 2$ and W_2 be the z-axis with dimension $q = 1$. Then, $W_1 + W_2 = \mathbb{R}^3$ and hence $dim(W_1 + W_2) = p + q = 3$.

Problem 5. (15 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. Let's denote $C(.)$, $N(.)$, $rank(.)$ as the column space, null space and rank of corresponding matrix, respectively. Find $C(\mathbf{A})$, $N(\mathbf{A})$, $dim(N(\mathbf{A}))$, $C(\mathbf{A}^T)$, $N(\mathbf{A}^T)$.

Solution:

The reduced row echelon form of $\mathbf{A}$ can be obtained as follows. Let R_i denotes the i-th row of the $\mathbf{A}$.

$$\xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

and by swapping the R_2 and R_3 we have

$$\xrightarrow{R_2 \to R_3 \text{ and } R_3 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix},$$

and then,

$$\xrightarrow{R_1 = R_1 - R_2 \text{ and } R_3 = R_3/2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Finally,

$$\xrightarrow{R_2 = R_2 - R_3} rref(\mathbf{A}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Clearly, the first, third and fourth columns are linearly independent and hence $rank(\mathbf{A}) = 3$ and $dim(N(\mathbf{A})) = 5 - 3 = 2$.

$$C(\mathbf{A}) = span \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Then,

$$\mathbf{A}\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 + x_5\mathbf{a}_5 = \mathbf{0}_5,$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ are the columns of $rref(\mathbf{A}^T)$ and $\mathbf{a}_1 = \mathbf{a}_2, \mathbf{a}_4 = \mathbf{a}_5$. Hence, we can write

$$(x_1 + x_2)\mathbf{a}_1 + x_3\mathbf{a}_3 + (x_4 + x_5)\mathbf{a}_4 = \mathbf{0}_3$$

and since $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_4$ are independent then

$$x_1 + x_2 = 0,$$
$$x_4 + x_5 = 0,$$

and therefore

$$N(\mathbf{A}) = span\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Besides,

$$\mathbf{A}^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and the reduced row echelon form of $\mathbf{A}^T$ can be obtained as follows. Let R_i denotes the i-th row of the $\mathbf{A}^T$.

$$\xrightarrow[R_4=R_4-R_1 \text{ and } R_5=R_5-R_1]{R_2=R_2-R_1\ ,\ R_3=R_3-R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix},$$

and by swapping the R_2 and R_4 we have

$$\xrightarrow{R_2 \to R_4 \text{ and } R_4 \to R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

and then,

$$\xrightarrow[R_2=R_2/2]{R_5=R_5-R_2\ ,\ R_1=R_1+R_2/2} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Finally,

$$\xrightarrow{R_1=R_1-R_3/2\ ,\ R_2=R_2-R_3/2} rref(\mathbf{A}^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $rank(\mathbf{A}^T) = 3$ and $dim(N(\mathbf{A}^T)) = 3 - 3 = 0$. Then,

$$C(\mathbf{A}^T) = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } N(\mathbf{A}^T) = \{\mathbf{0}_3\}.$$

Problem 6. (10 points) Let V be a finite dimensional vector space and $W_1, W_2 \subset V$ be two subspaces. Then *sum* of the two subspaces is defined as

$$S = W_1 + W_2 \triangleq \{\mathbf{w}_1 + \mathbf{w}_2 \,|\, \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}.$$

Then we define *direct sum* of the two subspaces as

$$\mathcal{S} \triangleq W_1 \oplus W_2,$$

if: (1) $\mathcal{S} = W_1 + W_2$ and (2) $W_1 \cap W_2 = \{\mathbf{0}\}$ (here $\oplus$ accounts for the direct sum).

(i) (5 points) Prove that $V = W_1 \oplus W_2$ if and only if any vector $\mathbf{v} \in V$ can be uniquely written as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in W_1$ and $\mathbf{v}_2 \in W_2$.

(ii) (5 points) Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution:

(i) ($\Rightarrow$) Suppose $V = W_1 \oplus W_2$ and let $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{v}_1, \mathbf{w}_1 \in W_1$ and $\mathbf{v}_2, \mathbf{w}_2 \in W_2$. Since W_1 and W_2 are subspaces, clearly $\mathbf{v}_1 - \mathbf{w}_1 \in W_1$ and $\mathbf{v}_2 - \mathbf{w}_2 \in W_2$. Then, $\mathbf{v}_1 - \mathbf{w}_1 = \mathbf{w}_2 - \mathbf{v}_2 \in W_2$. Since $W_1 \cap W_2 = \{\mathbf{0}\}$, then $\mathbf{v}_1 = \mathbf{w}_1$ and $\mathbf{v}_2 = \mathbf{w}_2$. Hence, $\mathbf{v} \in V$ can be written as the sum of two vectors in W_1 and W_2.
($\Leftarrow$) Suppose for any $\mathbf{v} \in V$ we have $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in W_1$ and $\mathbf{v}_2 \in W_2$. Let $V = W_1 + W_2$ where $W_1 \cap W_2 \neq \{\mathbf{0}\}$. Suppose by contradiction $\mathbf{w} \in W_1 \cap W_2$ and $\mathbf{w} \neq \mathbf{0}$. Then we can write $\mathbf{w} = \mathbf{w} + \mathbf{0}$ where $\mathbf{w} \in W_1$ and $\mathbf{0} \in W_2$ or $\mathbf{w} = \mathbf{0} + \mathbf{w}$ where $\mathbf{0} \in W_1$ and $\mathbf{w} \in W_2$. Hence, it is a clear contraction with the first assumption and hence $V = W_1 \oplus W_2$. $\square$

(ii) ($\Leftarrow$) If $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then clearly $W_1 \cup W_2 = W_1$ or W_2 whcih is a subspace in either case.
($\Rightarrow$) Assume by contradiction $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. This implies there are $\mathbf{w}_1 \in W_1$ but $\mathbf{w}_1 \notin W_2$ and $\mathbf{w}_2 \in W_2$ but $\mathbf{w}_2 \notin W_1$. Then, clearly $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 \cup W_2$ (i.e., $\mathbf{w}_1 + \mathbf{w}_2 \in W_1$ or W_2) since $W_1 \cup W_2$ is a subspace of V. Consider the case $\mathbf{w}_1 + \mathbf{w}_2 \in W_1$. Clearly, $(\mathbf{w}_1 + \mathbf{w}_2) - \mathbf{w}_1 \in W_1$ and hence $\mathbf{w}_2 \in W_1$. This is a clear contradiction to the assumption that $\mathbf{w}_2 \in W_2$ but $\mathbf{w}_2 \notin W_1$ and the proof is completed. $\square$

Problem 7. (10 points) Let $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 6 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ \beta \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 3 \\ 3\beta \\ 2\beta \end{bmatrix}$ where $\alpha \in \mathbb{R}$.

(i) (5 points) Find the values of β such that $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent.

(ii) (5 points) Find the values of β such that $\mathbf{w}_3 \in span\{\mathbf{w}_1, \mathbf{w}_2\}$.

Solution:

(i) For $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ to be linearly independent, we must have

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = 0,$$

if and only if $c_1 = c_2 = c_3 = 0$. Then,

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 3 & \beta & 3 & 0 \\ 9 & 3 & 3\beta & 0 \\ 6 & 2 & 2\beta & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 0 & \beta - 1 & 1 & 0 \\ 0 & 0 & 3\beta - 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which implies that $\beta \neq 1$ and $\beta \neq 2$.

(ii) We have $\mathbf{w}_3 \in span\{\mathbf{w}_1, \mathbf{w}_2\}$ if and only if $\mathbf{w}_3 = c_1\mathbf{w}_1 + c_2\mathbf{w}_2$ where $c_1, c_2 \in \mathbb{R}$. We can write

$$\left[\begin{array}{cc|c} 3 & 1 & 2 \\ 3 & \beta & 3 \\ 9 & 3 & 3\beta \\ 6 & 2 & 2\beta \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 2 \\ 0 & \beta - 1 & 1 \\ 0 & 0 & 3\beta - 6 \\ 0 & 0 & 0 \end{array}\right]$$

For $\beta = 2$ this system is consistent and hence $c_1 = 1/3$ and $c_2 = 1$.