EE 203001 SES #26

Markov matrices ; Fourier series

Markov matrices

Suppose we have a positive vector $\underline{u}_0 = \begin{bmatrix} a \\ 1-a \end{bmatrix}$

$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ (Markov matrix, col.s add to 1)

if $\underline{u}_1 = A\underline{u}_0$, $\underline{u}_2 = A\underline{u}_1$ ...

Q: What happens if we keep doing this?

$\underline{u}_1, \underline{u}_2, \dots$ converges to $\underline{u}_\infty$ (steady state)

For $\underline{u}_\infty$, $\underline{u}_\infty = A\underline{u}_\infty$

(multiplied by A does NOT change $\underline{u}_\infty$)

($\underline{u}_\infty$ is an eigenvector with $\lambda = 1$)

Def Markov matrix

A is a Markov matrix if:

1. Every entry of A is nonnegative
2. Every col. of A adds to 1

$\boxed{\text{Fact}}$

1. For nonnegative $\underline{u}_0$, $\underline{u}_1 = A\underline{u}_0$ is also nonnegative
2. If components of $\underline{u}_0$ add to 1, so do the components of $\underline{u}_1 = A\underline{u}_0$

Reason:

1. Trivial since both A & $\underline{u}_0$ are nonnegative
2. Components of $\underline{u}_0$ add to 1

$\Rightarrow [1, \cdots, 1]\,\underline{u}_0 = 1$

A is Markov $\Rightarrow$ every col. of A adds to 1 $\Rightarrow [1, \cdots, 1]\,A = [1, \cdots, 1]$

$\Rightarrow [1, \cdots, 1]\,\underbrace{A\,\underline{u}_0}_{(=\underline{u}_1)} = [1, \cdots, 1]\,\underline{u}_0 = 1$

$\Rightarrow$ Components of $\underline{u}_1$ add to 1

Note: same fact applies to

$\underline{u}_2 = A\underline{u}_1$, $\underline{u}_3 = A\underline{u}_2$, ...

$\Rightarrow$ every $\underline{u}_k = A^k\underline{u}_0$ is nonnegative with components adding to 1

($\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_k, \cdots$ are prob. vectors the limit $\underline{u}_\infty$ is also a prob. vector but we have to show that such limit exists)

Note: A^k is also a Markov matrix

$$([1,\cdots,1]A^k = [1,\cdots,1]AA^{k-1} = [1,\cdots,1]A^{k-1} = \cdots = [1,\cdots,1]A = 1)$$

Ex 1 (p.432)

Fraction of rental cars in Denver starts at 0.02 (outside is 0.98)

Every month: 80% of Denver cars stay in Denver (20% leave), 5% of outside cars comes in (95% stay outside)

$$\Rightarrow \begin{bmatrix} u_{Denver} \\ u_{outside} \end{bmatrix}_{t=k+1} = \underbrace{\begin{bmatrix} 0.8 & 0.05 \\ 0.2 & 0.95 \end{bmatrix}}_{A} \begin{bmatrix} u_{Denver} \\ u_{outside} \end{bmatrix}_{t=k}$$

$$\begin{bmatrix} u_{Denver} \\ u_{outside} \end{bmatrix}_{t=0} = \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{Denver} \\ u_{outside} \end{bmatrix}_{t=1} = \begin{bmatrix} 0.8 & 0.05 \\ 0.2 & 0.95 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 0.065 \\ 0.935 \end{bmatrix}$$

Q: What happens in the long run?

we are studying eqns: $\underline{u}_{k+1} = A\,\underline{u}_k$

$$\Rightarrow \underline{u}_k = A^k \underline{u}_0 = c_1\lambda_1^k \underline{x}_1 + \cdots + c_n\lambda_n^k \underline{x}_n$$

Need eigenvalues & eigenvectors to diagonalize A

$$|A - \lambda I| = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 0.75$$

$$\Rightarrow (A - I)\underline{x}_1 = \underline{0} \Rightarrow \underline{x}_1 = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$ (components add to 1)

$$(A - 0.75I)\underline{x}_2 = \underline{0} \Rightarrow \underline{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\underline{u}_0 = \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix} = c_1\underline{x}_1 + c_2\underline{x}_2 = 1\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} + 0.18\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \underline{u}_k = 1(1)^k \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} + 0.18(0.75)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(as $k \to \infty$) (steady state) (vanishing)

(eigenvector with $\lambda = 1$ is the steady-state)

(other eigenvector $\underline{x}_2$ disappears $\because |\lambda| < 1$)

(More steps we take, closer to $\underline{u}_\infty = (0.2, 0.8)$)

(True even when $\underline{u}_0 = (0, 1)$)

$\boxed{\text{Fact}}$ A is a positive Markov matrix ($a_{ij} > 0$), then $\lambda_1 = 1$ is larger than any other eigenvalues. The eigenvector $\underline{x}_1$ is the steady-state

$$\underline{u}_k = \underline{x}_1 + c_2(\lambda_2)^k \underline{x}_2 + \cdots + c_n(\lambda_n)^k \underline{x}_n$$

$\underline{u}_\infty = \underline{x}_1$ for any initial $\underline{u}_0$

Reason:

1. $\lambda = 1$ is an eigenvalue:

Every col. of $A - I$ adds to $1 - 1 = 0$

$\Rightarrow$ rows of $A - I$ add to the zero row

$\Rightarrow$ $A - I$ is singular

$\Rightarrow$ $|A - I| = 0 \Rightarrow \lambda = 1$

Alternative reason:

rows of $A - I$ add to the zero row

$\Rightarrow [1, \cdots, 1](A - I) = [0, \cdots, 0]$

$\Rightarrow (A^T - I)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \underline{0}$

$\Rightarrow \lambda = 1$ is an eigenvalue of A^T

$\Rightarrow \lambda = 1$ is an eigenvalue of A

$(|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I|$

$\Rightarrow A$ & A^T have same eigenvalues$)$

2. No eigenvalue can have $|\lambda| > 1$:

If there is any eigenvalue $|\lambda| > 1$

$\Rightarrow A^k$ will grow

But A^k is a Markov matrix

$\Rightarrow$ every col. of A^k adds to 1

$\Rightarrow$ no room to grow $\Rightarrow$ contradiction!

3. $c_1 = 1$ if components of $\underline{u}_0$ & $\underline{x}_1$ add to 1:

$[1, \cdots, 1] A \underline{x}_i = [1, \cdots, 1] \lambda_i \underline{x}_i$

$\Rightarrow [1, \cdots, 1] \underline{x}_i = \lambda_i [1, \cdots, 1] \underline{x}_i$

For λ_i, $i \geq 2$, $\lambda_i \neq 1 \Rightarrow [1, \cdots, 1] \underline{x}_i = 0$

$\underline{u}_0 = c_1 \underline{x}_1 + c_2 \underline{x}_2 + \cdots + c_n \underline{x}_n$

$\Rightarrow [1, \cdots, 1] \underline{u}_0 = c_1 [1, \cdots, 1] \underline{x}_1$

$\Rightarrow c_1 = 1$ if components of $\underline{u}_0$ & $\underline{x}_1$ add to 1

Note: In some applications, Markov matrices are defined differently: rows add up to 1 instead

(calculations are transpose of everything we've done here)

Fourier series & projections

Expansion with an orthonormal basis

If we have an orthonormal basis $\underline{q}_1, \underline{q}_2, \cdots, \underline{q}_n$, we can write any vector as

$$\underline{v} = x_1 \underline{q}_1 + \cdots + x_n \underline{q}_n$$

where

$$\underline{q}_i^T \underline{v} = x_1 \underline{q}_i^T \underline{q}_1 + \cdots + x_i + \cdots + x_n \underline{q}_i^T \underline{q}_n$$
$$= x_i \quad (\underline{q}_i^T \underline{q}_j = 0, \; i \neq j)$$

In terms of matrix:

$$[\underline{q}_1, \cdots, \underline{q}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{v}$$

$$\Rightarrow \quad Q \underline{x} = \underline{v} \Rightarrow \underline{x} = Q^{-1} \underline{v} = Q^T \underline{v}$$

$$\Rightarrow \quad x_i = \underline{q}_i^T \underline{v}$$

(key idea: express $\underline{v}$ = comb. of projection onto orthonormal basis vectors)

Fourier series

Same idea on functions!

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

(express $f(x)$ as comb. of projection onto trigonometric fcns)

(extend to infinite series)

vectors: fcns

basis: $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$

Q: What does orthogonal mean in this context?

Need to define inner product first

Vectors in R^n:

$$\underline{v}^T \underline{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

For fcns:

$$(f, g) = \int_0^{2\pi} f(x) g(x) \, dx$$

(integrate over $[0, 2\pi]$ since Fourier series are periodic, i.e., $f(x) = f(x+2\pi)$)

Chk orthogonality:

$$\int_0^{2\pi} \sin x \cos x \, dx = \frac{1}{2}(\sin x)^2 \Big|_0^{2\pi} = 0$$

$\vdots$ (inner product $= 0$)

Q: How to find Fourier coeff. $a_0, a_1, b_1, \cdots$?

a_0: average of $f(x)$

$$\left(\frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx = a_0 + \frac{1}{2\pi} \int_0^{2\pi} a_1 \cos x \, dx \right.$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} b_1 \sin x \, dx + \cdots = a_0)$$

a_1: $\int_0^{2\pi} f(x) \cos x \, dx$

$$= \int_0^{2\pi} (a_0 + a_1 \cos x + b_1 \sin x + \cdots) \cos x \, dx$$

$$= 0 + \int_0^{2\pi} a_1 \cos^2 x \, dx + 0 + \cdots$$

$$= \int_0^{2\pi} a_1 \frac{1+\cos 2x}{2} \, dx = \pi a_1$$

$$\Rightarrow a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos x \, dx$$

Similarly,

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin kx \, dx$$

(read Ex3. p.449)