EE 203001 SBS #25

Differential equs & e^{At}

Scalar ODE (one equ)

$\frac{du}{dt} = \lambda u$ has sol.s $u(t) = c e^{\lambda t}$

at $t=0$, $u(0) = c$

$\Rightarrow u(t) = u(0) e^{\lambda t}$

Q: How about n equs ?

Start with 2 equs

$$\frac{du_1}{dt} = -u_1 + 2u_2$$
$$\frac{du_2}{dt} = u_1 - 2u_2$$

describe how values of var.s u_1 & u_2 affect each other over time

Just as we apply linear algebra to solve difference equs, we can use it to solve differential equs

Differential equs: $\frac{d\underline{u}}{dt} = A\underline{u}$

Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ starting from $\underline{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

We can guess that $\underline{u} = e^{\lambda t}\underline{x}$ is a sol. when $A\underline{x} = \lambda\underline{x}$

(eigenvalue & eigenvectors)

Q: Is this true ?

$$\left.\begin{aligned} \frac{d\underline{u}}{dt} &= \lambda e^{\lambda t}\underline{x} \\ A\underline{u} &= e^{\lambda t}A\underline{x} = \lambda e^{\lambda t}\underline{x} \end{aligned}\right\} \Rightarrow \frac{d\underline{u}}{dt} = A\underline{u} \quad (\checkmark)$$

Back to example:

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} : \text{singular} \Rightarrow \lambda_1 = 0$$

$$\text{trace}(A) = -3 = \lambda_1 + \lambda_2 \Rightarrow \lambda_2 = -3$$

Find corr. eigenvectors:

$$A\underline{x}_1 = \underline{0} \Rightarrow \underline{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A+3I)\underline{x}_2 = \underline{0} \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\underline{x}_2 = \underline{0}$$

$$\Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \underline{u}_1(t) = e^{\lambda_1 t}\underline{x}_1 = e^{0t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\underline{u}_2(t) = e^{\lambda_2 t}\underline{x}_2 = e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(pure sol.s)

Complete sol:

$$\underline{u}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(steady state sol.) (decays to zero as $t \to \infty$)

$$\underline{u}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = c_2 = 1/3$$

$$\underline{u}(t) = 1/3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1/3\, e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underline{u}(\infty) = 1/3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ (steady state sol.)

Summary:

Step 1: write $\underline{u}(0)$ as combination $c_1 \underline{x}_1 + \cdots + c_n \underline{x}_n$ of eigenvectors of A

Step 2: Multiply each eigenvector $\underline{x}_i$ by $e^{\lambda_i t}$ (pure sol.)

Step 3: Complete sol. is a comb. of pure sol.s

$$\underline{u}(t) = c_1 e^{\lambda_1 t} \underline{x}_1 + \cdots + c_n e^{\lambda_n t} \underline{x}_n$$

(Analogy: $c_1 \lambda_1^k \underline{x}_1 + \cdots + c_n \lambda_n^k \underline{x}_n$ sol. to diff. eqns)

Stability

Not all systems have a steady state

$\Rightarrow$ eigenvalues of A tell us what to expect

1. Stability: $\underline{u}(t) \to 0$ when $\mathrm{Re}(\lambda) < 0$
2. Steady state: One eigenvalue is 0
   all other eigenvalues have negative real parts
3. Blow up: $\mathrm{Re}(\lambda) > 0$ for any λ

For 2x2

$\boxed{\text{Fact}}$ For 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

System is stable if $\mathrm{Re}(\lambda) < 0$

$\Leftrightarrow$ trace $T = a+d < 0$ $\quad (\lambda_1 + \lambda_2 < 0)$

det $D = ad - bc > 0$ $\quad (\lambda_1 \lambda_2 > 0)$

Reason:

"$\Rightarrow$" If λ's are real & negative

sum $= T < 0$, $\lambda_1 \lambda_2 = D > 0$

"$\Leftarrow$" If $D > 0$, λ_1, λ_2 has same sign

If $T < 0$, both $\lambda_1, \lambda_2 < 0$

Complex λ's:

$\lambda_1 = r + is$, $\lambda_2 = r - is$

(otherwise T is not real)

$D = \lambda_1 \lambda_2 = r^2 + s^2 > 0$

$T = \lambda_1 + \lambda_2 = 2r$

so if $T < 0 \Rightarrow Re(\lambda_1), Re(\lambda_2) < 0$

if $r < 0 \Rightarrow T < 0$

<u>Matrix exponential</u> : e^{At}

Q: What does e^{At} mean if A is a matrix?

Recall: for a real number

$$e^{x} = 1 + x + \frac{x^2}{2!} + \cdots$$

Define e^{At} using the same formula

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

Note 1: derivative of e^{At}

$$\frac{de^{At}}{dt} = A + A^2 t + \frac{1}{2} A^3 t^2 + \cdots = Ae^{At}$$

Note 2: eigenvalues of e^{At}

$$\begin{aligned} e^{At}\underline{x} &= \left(I + At + \frac{(At)^2}{2!} + \cdots\right)\underline{x} \\ &= \left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots\right)\underline{x} \\ &= e^{\lambda t}\underline{x} \Rightarrow \text{eigenvalues} = e^{\lambda t} \end{aligned}$$

Note 3: $e^{At} = S e^{\Lambda t} S^{-1}$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \cdots$$

$$= SS^{-1} + S\Lambda S^{-1} + S\left(\frac{\Lambda^2 t^2}{2!}\right)S^{-1} + \cdots$$

$$= S e^{\Lambda t} S^{-1}$$

$$= S \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} S^{-1}$$

(easier way to compute e^{At})

<u>Alternative way to solve</u>: $\frac{d\underline{u}}{dt} = A\underline{u}$

Note: $\frac{d\underline{u}}{dt} = A\underline{u}$

(couples the pure sol.s)

Let $\underline{u} = S\underline{v}$ (S: matrix of eigenvectors)

$\Rightarrow S\frac{d\underline{v}}{dt} = AS\underline{v}$

$\Rightarrow \frac{d\underline{v}}{dt} = S^{-1}AS\underline{v} = \Lambda\underline{v}$

This diagonalize the system:

$$\frac{dv_i}{dt} = \lambda_i v_i \quad , i = 1, \cdots, n$$

General sol:

$$\underline{v}(t) = e^{\Lambda t}\underline{v}(0)$$

$$\Rightarrow S^{-1}\underline{u}(t) = e^{\Lambda t} S^{-1}\underline{u}(0)$$

$$\Rightarrow \underline{u}(t) = S e^{\Lambda t} S^{-1} \underline{u}(0) = e^{At}\underline{u}(0)$$

Recall:

$$\underline{u}(0) = c_1 \underline{x}_1 + c_2 \underline{x}_2 + \cdots + c_n \underline{x}_n$$

$$= S \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow S^{-1}\underline{u}(0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\Rightarrow e^{At}\underline{u}(0) = S e^{\Lambda t} S^{-1} \underline{u}(0)$$

$$= \begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= c_1 e^{\lambda_1 t} \underline{x}_1 + \cdots + c_n e^{\lambda_n t} \underline{x}_n$$

(Same as before)

(read Ex 6, p.321)

Note 1: e^{At} always has inverse e^{-At}

Reason: $e^{At} = S e^{\Lambda t} S^{-1}$

$$\Rightarrow (e^{At})^{-1} = S (e^{\Lambda t})^{-1} S^{-1}$$

$$= S e^{-\Lambda t} S^{-1} = e^{-At}$$

($-A$ & A have same eigenvectors and eigenvalues with a minus sign)

Note 2: The eigenvalues of e^{At} are always $e^{\lambda t}$

Reason:

$$e^{At} = S e^{\Lambda t} S^{-1}$$

$$\Rightarrow e^{At} S = S e^{\Lambda t}$$

$$\Rightarrow \text{eigenvalues } e^{\lambda_1 t} \ldots e^{\lambda_n t}$$

Note 3: When A is skew-symmetric e^{At} is orthogonal ($A^T = -A$)

(Inverse = tranpose = e^{-At})

Reason:

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \cdots$$

$$\Rightarrow (e^{At})^T = I + A^T t + \frac{1}{2!}(A^T t)^2 + \cdots$$

$$= I + (-A)t + \frac{1}{2!}(-At)^2 + \cdots$$

$$= e^{-At}$$

(Read Ex. 5. p. 320)

Second order

$$y'' + by' + ky = 0$$

guess sol. $y = e^{\lambda t}$

$$\Rightarrow (\lambda^2 + b\lambda + k) e^{\lambda t} = 0$$

or we can change it into a 2×2 first-order system

Let $\underline{u} = \begin{bmatrix} y' \\ y \end{bmatrix}$

$$\Rightarrow \underline{u}' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

$$\Rightarrow \underline{u}' = A\,\underline{u}$$

Find eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} -b-\lambda & -k \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + b\lambda + k = 0$$

(same as before)

eigenvectors: $\underline{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

$$\Rightarrow \underline{u}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

kth order eqn:

we get a $k \times k$ matrix:

coeff. of eqn in 1st row

& 1's in the diagonal below that

& the rest of entries $= 0$

(Read Ex 4, p. 320)