Diagonalization & powers of A

We learned eigenvalues & eigenvectors

$\Rightarrow$ We can diagonalize a matrix A using eigenvectors if A has n indep. eigenvectors

Diagonalize a matrix: $S^{-1}AS = \Lambda$

$\boxed{\text{Fact}}$ Suppose $n \times n$ matrix A has n indep. eigenvectors $\underline{x}_1, \cdots, \underline{x}_n$. Put them into col.s of an eigenvector matrix S. Then, $S^{-1}AS$ is the eigenvalue matrix Λ, i.e.,

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Reason:

$$AS = A\begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \underline{x}_1 & \lambda_2 \underline{x}_2, & \cdots, & \lambda_n \underline{x}_n \end{bmatrix}$$

$$= S \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = S\Lambda$$

Since col.s of S are indep.

$\Rightarrow$ S is invertible $\Rightarrow$ S^{-1} exists

$$AS = S\Lambda \Rightarrow S^{-1}AS = \Lambda$$

or $A = S\Lambda S^{-1}$

Note: A can be diagonalize since S has an inverse

$\Rightarrow$ Without n indep. eigenvectors, we cannot diagonalize

Powers of A

Q: What are the eigenvalues & eigenvectors of A^2?

If $A\underline{x} = \lambda \underline{x}$

then $A(A\underline{x}) = \lambda A \underline{x}$

$\Rightarrow A^2 \underline{x} = \lambda^2 \underline{x}$

(Eigenvalues of A^2 are squares of eigenvalues of A)

(Eigenvectors of A^2 are the same as eigenvectors of A)

Alternatively,

$$A = S \Lambda S^{-1}$$

$$\Rightarrow A^2 = S \Lambda S^{-1} S \Lambda S^{-1} = S \Lambda^2 S^{-1}$$

Similarly,

$$A^k = S \Lambda^k S^{-1}$$

(eigenvalues raised to the k^{th} power)

(eigenvectors stay the same)

Note 1: we can multiply eigenvectors by nonzero constants

($A\underline{x} = \lambda \underline{x} \Rightarrow A(c\underline{x}) = \lambda(c\underline{x})$

$\Rightarrow c\underline{x}$ is also an eigenvector)

Note 2: there is no connection between invertibility & diagonalizability

- Invertibility: whether eigenvalues $\lambda = 0$ or $\lambda \neq 0$

($\lambda = 0 \Rightarrow A\underline{x} = \underline{0}$ for some nonzero $\underline{x}$

$\Rightarrow A$ is singular)

- Diagonalizability: whether we have n indep. eigenvectors

(A has indep. col. vectors $\Leftrightarrow$
A is invertible)

(A has indep. eigenvectors $\Leftrightarrow$
A is diagonalizable)

Note 3: Suppose all eigenvalues $\lambda_1, \ldots, \lambda_n$ are different $\Rightarrow$ eigenvectors $\underline{x}_1, \ldots, \underline{x}_n$ are indep. $\Rightarrow$ A can be diagonalized

(Any matrix with no repeated eigenvalues can be diagonalized)

Reason: chk 2×2 case

Suppose $c_1 \underline{x}_1 + c_2 \underline{x}_2 = \underline{0}$ ($\underline{x}_1, \underline{x}_2$: eigenvectors)

multiplied by A $\Rightarrow \quad c_1 A \underline{x}_1 + c_2 A \underline{x}_2 = \underline{0}$

$\Rightarrow c_1 \lambda_1 \underline{x}_1 + c_2 \lambda_2 \underline{x}_2 = \underline{0}$

multiplied by λ_2 $-)$ $\Rightarrow c_1 \lambda_2 \underline{x}_1 + c_2 \lambda_2 \underline{x}_2 = \underline{0}$

$$c_1 (\lambda_1 - \lambda_2) \underline{x}_1 = \underline{0}$$

$$\Rightarrow c_1 = 0 \text{ if } \lambda_1 \neq \lambda_2$$

Similarly, $c_2 = 0$ if $\lambda_2 \neq \lambda_1$

so $\underline{x}_1, \underline{x}_2$ are lin. indep.

Extend to $n \times n$ matrix,

Suppose $c_1 \underline{x}_1 + c_2 \underline{x}_2 + \cdots + c_n \underline{x}_n = 0$

($\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$: eigenvectors)

multiplied by $(A - \lambda_n)$ $\quad c_1(\lambda_1 - \lambda_n)\underline{x}_1 + \cdots + c_{n-1}(\lambda_{n-1} - \lambda_n)\underline{x}_{n-1} = \underline{0}$

multiplied by $(A - \lambda_{n-1})$ $\quad c_1(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n-1})\underline{x}_1 + \cdots + c_{n-2}(\lambda_{n-2} - \lambda_n)(\lambda_{n-2} - \lambda_{n-1})\underline{x}_{n-2} = \underline{0}$

$\vdots$

multiplied by $(A - \lambda_2)$ $\quad c_1(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n-1}) \cdots (\lambda_1 - \lambda_2)\underline{x}_1 = \underline{0}$

$\Rightarrow c_1 = 0$ since λ_i's are diff.

Similarly, $c_2 = c_3 = \cdots = c_n = 0$

$\Rightarrow \underline{x}_1, \ldots, \underline{x}_n$ are lin. indep. ∇

Ex: powers of $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 0.5$

$(A - \lambda_1 I)\underline{x}_1 = \underline{0} \Rightarrow \underline{x}_1 = (0.6, 0.4)$

$(A - \lambda_2 I)\underline{x}_2 = \underline{0} \Rightarrow \underline{x}_2 = (1, -1)$

$A = S \Lambda S^{-1}$

$$\Rightarrow \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

Same S for A^2 $\Rightarrow A^2 = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^2 & 0 \\ 0 & (0.5)^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$

Same S for A^k $\Rightarrow A^k = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (0.5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$

Limit $k \to \infty$ $\Rightarrow A^\infty = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$

(steady state)

$\boxed{\text{Fact}}$ If A has n indep. eigenvectors with eigenvalues λ_i, then

$A^k \to 0$ as $k \to \infty$ iff all $|\lambda_i| < 1$

(zero matrix)

Repeated eigenvalues

If A has repeated eigenvalues, it may or may not have indep. eigenvectors

Ex 1: $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \lambda_1 = \lambda_2 = 1$

$(A - \lambda I)\underline{x} = \underline{0} \Rightarrow$ any $\underline{x}$ would work

$\Rightarrow N(A - I)$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ indep. eigenvectors

Ex 2: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\lambda_1 = \lambda_2 = 2$

$(A - \lambda I)\underline{x} = \underline{0} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\underline{x} = \underline{0}$

$\Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ($N(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$ has dim = 1)

$\Rightarrow$ only one eigenvector

$\Rightarrow$ no indep. eigenvectors

Difference eqn $\underline{u}_{k+1} = A\underline{u}_k$

Starting with $\underline{u}_0$

$\underline{u}_{k+1} = A\underline{u}_k$ is a first-order difference eqn

Sol: $\underline{u}_k = A^k \underline{u}_0$

Write $\underline{u}_0$ as comb. of eigenvectors of A, i.e.,

$$\underline{u}_0 = c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots + c_n\underline{x}_n$$

$$= S\underline{c}$$

then

$$A\underline{u}_0 = c_1\lambda_1\underline{x}_1 + c_2\lambda_2\underline{x}_2 + \cdots + c_n\lambda_n\underline{x}_n$$

$$= S\Lambda\underline{c}$$

and

$$A^k\underline{u}_0 = c_1\lambda_1^k\underline{x}_1 + c_2\lambda_2^k\underline{x}_2 + \cdots + c_n\lambda_n^k\underline{x}_n$$

$$= S\Lambda^k\underline{c}$$

$$\Rightarrow \underline{u}_k = A^k\underline{u}_0 = c_1\lambda_1^k\underline{x}_1 + \cdots + c_n\lambda_n^k\underline{x}_n$$

Fibonacci sequence

The sequence : 0, 1, 1, 2, 3, 5, 8, 13, ...

$F_{k+2} = F_{k+1} + F_k$ (2nd order diff. eqn)

Q: How do we solve a 2nd order eqn?

Convert it into 1st-order eqn

Let $\underline{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$, then

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{k+1} = F_{k+1}$$

equivalent to

$$\underline{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underline{u}_k$$

Step I : Find eigenvalues & eigenvectors

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 1 = 0$$

$$\Rightarrow \lambda_1 = \tfrac{1}{2}(1+\sqrt{5}), \quad \lambda_2 = \tfrac{1}{2}(1-\sqrt{5})$$

Since $(A - \lambda I)\underline{x} = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \underline{x} = \underline{0}$

is $\underline{x} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \Rightarrow \underline{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \underline{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

Step 2 : Find $\underline{u}_0 = c_1 \underline{x}_1 + c_2 \underline{x}_2$

$$\underline{u}_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\Rightarrow c_1 = -c_2 = \frac{1}{\sqrt{5}}$$

Step 3:

$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \underline{u}_k = c_1 \lambda_1^k \underline{x}_1 + c_2 \lambda_2^k \underline{x}_2$$

$$\Rightarrow F_k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^k - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^k$$

(Using eigenvalues & eigenvectors, we obtain closed-form expression for Fibonacci sequence)

Summary: When a sequence evolves over time following 1st order diff. eqn $\Rightarrow$ eigenvalues of the system matrix determine long term behavior of the series