EE 203001 SES # 23

## Eigenvalues & Eigenvectors

Eigenvalues : special numbers associated with a matrix

Eigenvectors : special vectors

## Q : How special ?

Almost all vectors change direction when multiplied by $A$ but

Eigenvectors $\underline{x}$ are in the same direction as $A\underline{x}$

$\boxed{\text{Def}}$ For an eigenvector of $A$ (non-zero)

$$A\underline{x} = \lambda \underline{x} \quad , \lambda : \text{eigenvalue}$$

( $\lambda$ tells whether the special vector $\underline{x}$ is stretched or shrunk or reversed or left unchanged )

## Eigenvalue 0

If the eigenvalue $\lambda = 0$ , then

$$A\underline{x} = 0\underline{x} = \underline{0} \Rightarrow \underline{x} \text{ in nullspace of } A$$

$\Rightarrow$ vectors of eigenvalue 0 makes up $N(A)$

If $A$ is singular, then $\lambda = 0$ is an eigenvalue of $A$

(Otherwise, $A\underline{x} = 0\,\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0}$)

## Projection matrix $P$

Suppose $P$: projection onto a plane

For any vector on the plane, we have

$P\underline{x}_1 = \underline{x}_1 \Rightarrow \underline{x}_1$ is an eigenvector with eigenvalue $1$

A vector $\underline{x}_2$ perpendicular to the plane

$P\underline{x}_2 = \underline{0} \Rightarrow \underline{x}_2$ is an eigenvector with eigenvalue $0$

(nonzero vector $\underline{x}_2 \in N(A) \Rightarrow A$ singular)

The eigenvectors of $P$ spans the entire space (NOT true for any matrix)

Ex: $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

$\lambda = 1 \Rightarrow P\underline{x} = \underline{x} \Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 0 \Rightarrow P\underline{x} = \underline{0} \Rightarrow \underline{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Note: Since $P = P^T$, eigenvectors are perpendicular (will prove this later)

Ex: The reflection matrix

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ has eigenvalues $1$ & $-1$

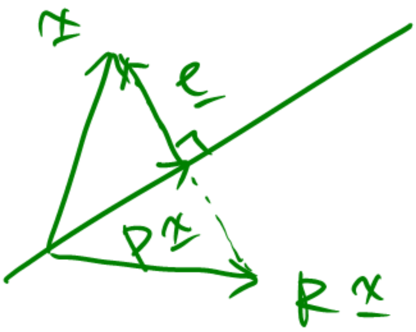
Recall: Eigenvectors for $P$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$R\underline{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 1$$

$$R\underline{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda = -1$$

$\Rightarrow$ same eigenvectors as $P$

Why? $R = 2P - I$




$$\underline{e} = (I - P)\underline{x}$$

$$R\underline{x} = \underline{x} - 2\underline{e}$$

$$= \underline{x} - 2(I-P)\underline{x}$$

$$= (2P - I)\underline{x}$$

If $\underline{x}$ is an eigenvector of $P$, then

$$P\underline{x} = \lambda\underline{x} \Rightarrow 2P\underline{x} = 2\lambda\underline{x}$$

$$- ) \quad I\underline{x} = \underline{x}$$

$$(2P - I)\underline{x} = (2\lambda - 1)\underline{x}$$

$$\Rightarrow R\underline{x} = (2\lambda - 1)\underline{x}$$

So same eigenvector for $R$ but eigenvalue: $\lambda \to 2\lambda - 1$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: $2(1) - 1 = 1$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$: $2(0) - 1 = -1$

# The eqn for Eigenvalues

An $n \times n$ matrix will have $n$ eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

Solve $A\underline{x} = \lambda \underline{x}$ to obtain eigenvalues & eigenvectors

$$\Rightarrow (A - \lambda I)\underline{x} = \underline{0}$$

In order for $\underline{x}$ to be an eigenvector, $A - \lambda I$ must be singular

$$\Rightarrow \det(A - \lambda I) = 0$$ (characteristic polynomial)

(involves ONLY $\lambda$, not $\underline{x}$)

# To obtain Eigenvectors

For each eigenvalue $\lambda$, solve

$(A - \lambda I)\underline{x} = \underline{0}$ or $A\underline{x} = \lambda \underline{x}$

(in nullspace of $A - \lambda I$)

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (singular)

When $A$ is singular, $\lambda = 0$ is one of eigenvalues since

$$A\underline{x} = 0\underline{x} = \underline{0} \text{ has sol.s}$$

vectors in $N(A)$ are eigenvectors

By eigenvalue eqn,

$$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

$$= (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda-5) = 0$$

$\Rightarrow \lambda = 0$ or $\lambda = 5$

(as expected)

Now, find eigenvectors

$$(A - 0I)\underline{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5\lambda)\underline{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5$$

(Matrix $A - 0I$ & $A - 5I$ are singular since $\lambda = 0$, $\lambda = 5$ are eigenvalues $(-2, 1)$, $(1, 2)$ are in the nullspaces)

Note: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has same eigenvector as $B = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$

$$( A\underline{x} = (B+I)\underline{x} = \lambda\underline{x} + \underline{x} = (\lambda+1)\underline{x}$$

$\Rightarrow$ eigenvalues of $A$ are one plus eigenvalues of $B$ but eigenvectors stay the same)

Bad news:

Elimination does NOT preserve $\lambda$'s

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has $\lambda = 0$, $\lambda = 5$

$U = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ has $\lambda = 0$, $\lambda = 1$

$\boxed{\text{Fact}}$ Eigenvalues of $U$ sit on its diagonal (pivots)

Recall: $\det U = u_{11} \cdots u_{nn}$

so $\det(U - \lambda I) = (u_{11} - \lambda) \cdots (u_{nn} - \lambda)$
$= 0$

$\Rightarrow \lambda = u_{11}, \lambda = u_{22}, \cdots, \lambda = u_{nn}$

Eigenvalues are changed during row operations!

Good news: when $A$ is $n \times n$,

(1) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$
$= \text{trace } A$

(For $2 \times 2$: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 - (a+d)\lambda + ad - bc$

$= \lambda^2 - (\text{trace } A)\lambda + \det A\ )$

(2) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

$\Big( \det(A - \lambda I) = \underbrace{(\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)}_{\text{polynomial of degree } n} = 0$

Let $\lambda = 0$, we have

$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n\ \Big)$

A caution:

If $A\underline{x} = \lambda \underline{x}$, $B\underline{x} = \alpha \underline{x}$

$\Rightarrow (A+B)\underline{x} = (\lambda + \alpha)\underline{x}$

So $A+B$ has eigenvalue $\lambda + \alpha$?

NOT really!

ONLY true when $A$ & $B$ have the same eigenvectors

Similarly, eigenvalues of

$$AB \neq \lambda(A)\lambda(B)$$

<u>Complex eigenvalues</u>

The matrix $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates every vector by $90^\circ$

trace $= 0 = \lambda_1 + \lambda_2$, $\det = 1 = \lambda_1 \lambda_2$

The only real eigenvector is $\underline{0}$ since any other vector changes direction when multiplied by $Q$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2+1 = 0$$

$\Rightarrow \lambda = i, -i$

Note: If $a+bi$ is an eigenvalue

$\Rightarrow a-bi$ is also "

Note: Symmetric matrices have Real eigenvalues

anti-symmetric " " Imaginary

" ($A^T = -A$, like $Q$)

## Triangular matrix & repeated eigenvalues

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \quad \lambda_1 = 3, \ \lambda_2 = 3$$

To find eigenvectors,

$$(A-3I)\underline{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} = \underline{0}$$

$\Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, there is NO indep. eigenvector $\underline{x}_2$