EE 203001 SBS # 22

Cramer's rule, inverse matrix, and volume

Many applications of det. Let's see how it is used!

Formula for A^{-1}

For 2x2:

we know $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\det A}} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\text{involves cofactors of } A}$

($C_{11} = \det(d) = d$, $C_{12} = -c$, $C_{21} = -b$

$C_{22} = a$ $\Rightarrow$ cofactor matrix $C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$)

Guess A^{-1} for general $n \times n$ matrix:

$$A^{-1} = \frac{1}{\det A} C^T \rightarrow \text{(product of } n-1 \text{ entries)}$$

(product of n entries)

(Now, it is possible that A^{-1} cancels with A)

(For 2x2, $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

(Much easier to see from this than elimination) pf of inverse formula:

Same as proving $AC^T = (\det A) I$

$$D = AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

$$d_{11} = \sum_{j=1}^{n} a_{1j} C_{1j} = \det A$$

$$\vdots \qquad\qquad \vdots$$

$$d_{nn} = \sum_{j=1}^{n} a_{nj} C_{nj} = \det A$$

Next, we want to show that all off-diagonal terms are zero

Say, row 2 of A & row 1 of C

(col 1 of C^T)

$$d_{21} = a_{21}C_{11} + a_{22}C_{12} + \cdots + a_{2n}C_{1n}$$

This is cofactor rule of a new matrix A^{21}

$$A' = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Obviously, $\det A' = 0$

In general,

$$d_{ij} = a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

det of A^{ij} (replace jth row of A by

i th row of A)

$\Rightarrow \det A^{ij} = 0$ for all $i \neq j$

$\Rightarrow A C^T = (\det A) I \Rightarrow A^{-1} = \frac{1}{\det A} C^T$

(This formula helps answer how inverse changes when the matrix changes)

Cramer's rule for $\underline{x} = A^{-1}\underline{b}$

If A is nonsingular & $A\underline{x} = \underline{b}$, then

$\underline{x} = A^{-1}\underline{b}$

Applying inverse formula $A^{-1} = C^T / \det A$

$\Rightarrow \underline{x} = C^T \underline{b} / \det A$

$\Rightarrow x_j = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$

$= \det B_j / \det A$

where we get B_j from A by replacing the jth col. from $\underline{b}$

(Usually less efficient than Elimination but more insights)

$|\det A|$ = volume of box

Claim: $|\det A|$ = volume of box whose edges are the row vectors of A (or col. vector since $\det A = \det A^T$)

pf: Show that volume of box satisfies property 1-3 of $|\det A|$

property 1: If $A = I$, the box is a unit cube $\Rightarrow$ vol. $= 1 = |\det I|$

(If $A = Q$, the box is a unit cube with diff. orientation & vol. $= 1 = |\det Q|$)

($\because Q$ is an orthogonal matrix $\Rightarrow Q^TQ = I$ $\Rightarrow (\det Q)^2 = 1 \Rightarrow \det Q = \pm 1$)

property 2: exchanging two rows of A does NOT change the vol. & $|\det A|$

property 3:

Chk 2x2 first:

$$\begin{vmatrix} t x_1 & t y_1 \\ x_2 & y_2 \end{vmatrix} = t \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \ ?$$

$$\begin{vmatrix} x_1 + x_1' & y_1 + y_1' \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_1' & y_1' \\ x_2 & y_2 \end{vmatrix} \ ?$$

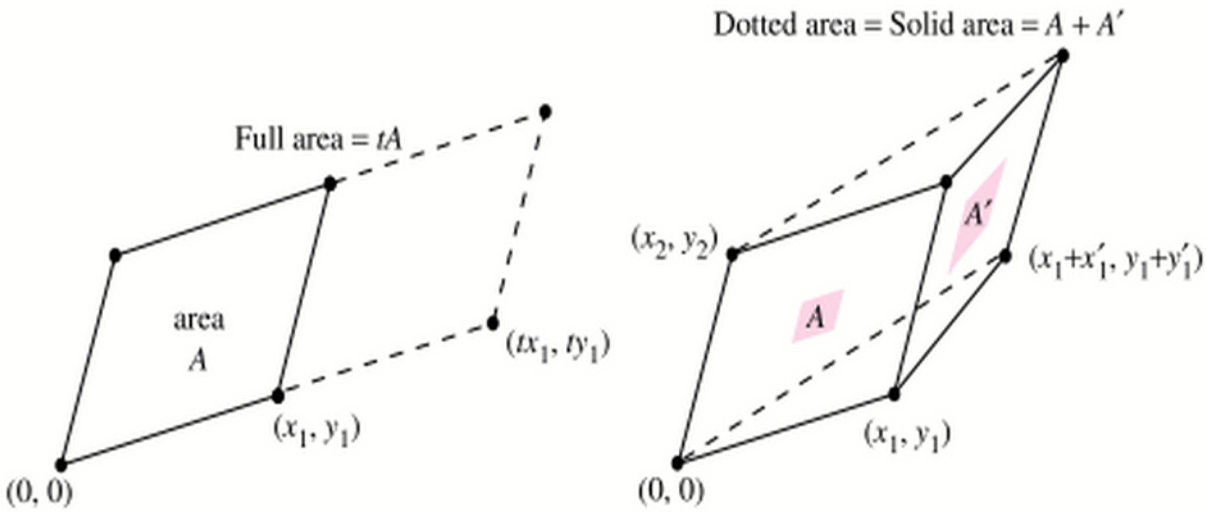


Figure 36: Areas obey the rule of linearity (keeping the side (x_2, y_2) constant).

Can be generalized to n dim box, e.g., 3×3

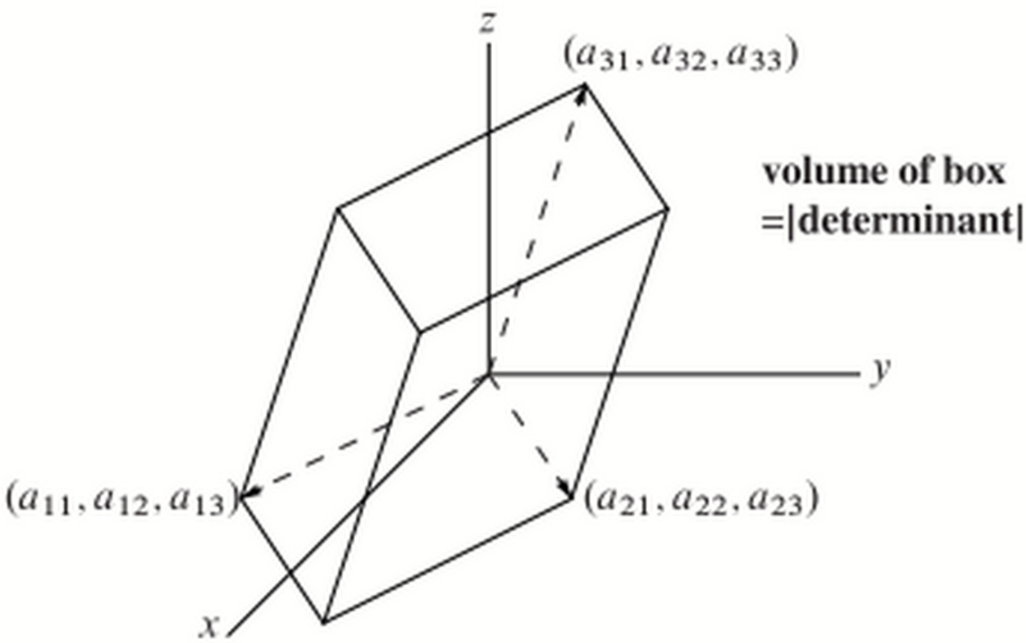


Figure 37: Three-dimensional box formed from the three rows of A.

Interesting to see: (not nec. for our proof)

If two edges of a box are equal, the box flatten out ⇒ vol. = 0 (property 4)

Important note:

If you know the corners of a box, then computing vol. is as easy as computing det

Ex:

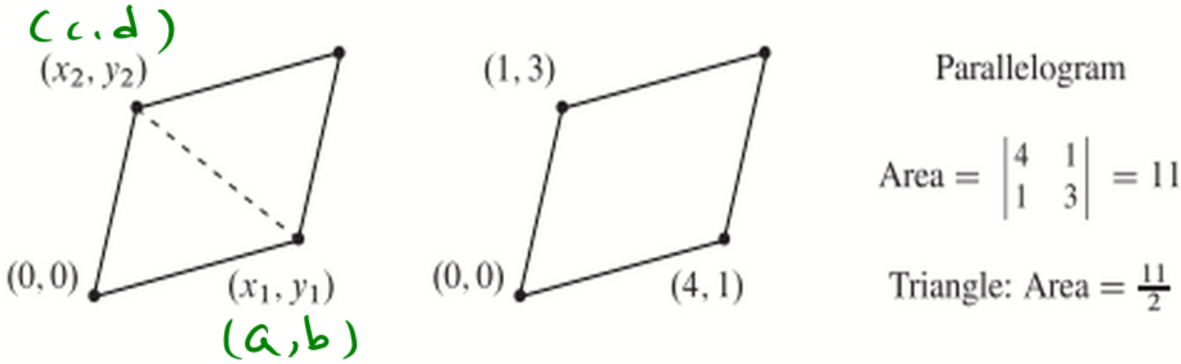


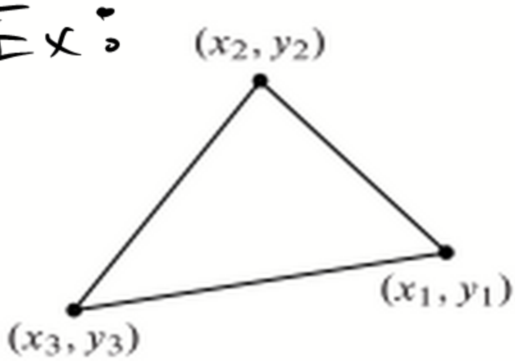
Figure 35: A triangle is half of a parallelogram. Area is half of a determinant.

area of parallelogram $= \left| \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right|$

$= |ad - bc|$

area of triangle $= \frac{1}{2} |ad - bc|$

Ex:



general triangle

$$\text{Area} = \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

$$(\because \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2-x_1 & y_2-y_1 & 0 \\ x_3-x_1 & y_3-y_1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}$$ (shift (x_1, y_1) to origin))