EE 203001 SES # 21

Determinant formulas & cofactors

We learned properties of det. Now, we are ready to obtain formulas for det:

1. Products of pivots
2. The "big formula"
3. Cofactors

Products of pivots (use Elimination)

Recall from SES-20,

$PA = LU \Rightarrow (\det P)(\det A) = (\det L)(\det U)$

$\Rightarrow \pm 1 (\det A) = 1 \cdot d_1 d_2 \cdots d_n$

$\Rightarrow \det A = \pm d_1 \cdots d_n$

(for invertible A)

For singular A, $\det A = 0 \because \det U = 0$
(zero rows in U)

The big formula

2×2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

break $[a\ b]$ into simple rows

$[a\ b] = [a\ 0] + [0\ b]$

break $[c \;\; d]$ into simple rows

$$[c \;\; d] = [c \;\; 0] + [0 \;\; d]$$

Now apply linearity in rows

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{3(b)}{=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$ (row 1 with row 2 fixed)

$$\overset{3(b)}{=} \underbrace{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}}_{=0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}_{=0}$$

(row 2 with row 1 fixed)

$$\overset{3(a)}{=} ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$\overset{1 \& 2}{=} ad - bc$$

(# of terms $= 2^2 = 4$, # of nonzero terms $= 2! = 2$)

3×3:

break each row to simple rows

e.g., $[a_{11} \;\; a_{12} \;\; a_{13}] = [a_{11} \;\; 0 \;\; 0] + [0 \;\; a_{12} \;\; 0] + [0 \;\; 0 \;\; a_{13}]$ (3 choices)

Same for row 2 & row 3
(3 choices) (3 choices)

$\Rightarrow$ a total of 3^3 simple det

But many zero det !

If a col. choice is repeated, then the simple det $= 0$

e.g., $[a_{11}\ 0\ 0]$ $[a_{21}\ 0\ 0]$

$$\Rightarrow \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ \times & \times & \times \end{vmatrix} = 0$$

$\Rightarrow$ nonzero terms only comes from diff. col.s (3! ways to order col.s)

$$\Rightarrow \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underset{(1,2,3)}{\begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix}} + \underset{(1,3,2)}{\begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix}} + \underset{(2,1,3)}{\begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix}} +$$

$$\underset{(2,3,1)}{\begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix}} + \underset{(3,1,2)}{\begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}} + \underset{(3,2,1)}{\begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} +$$

$$a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} +$$

$$a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
$$+ a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

An easy way to remember:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31}$$
$$+ a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31}$$
$$- a_{23}a_{32}a_{11} - a_{12}a_{21}a_{33}$$

But this only works for 2x2 & 3x3

NOT for higher n

(e.g., for 4x4, this only produces 8 products but we actually have $4! = 24$ products)

<u>In general</u> ($n \times n$)

There are $n!$ col. ordering

Let $(\alpha, \beta, \cdots, \omega)$ be one possible ordering

$\Rightarrow$ this simple det $= \pm\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$

(± 1: determined by $P = (\alpha, \beta, \cdots, \omega)$)

(e.g., $P = (1,2,3) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

$P = (3,1,2) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$

$P = (2,1,3) = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} \cdots$)

then

$$\det A = \sum_{n!\ \text{terms}} (\det P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$ (the "big formula")

where $(\alpha, \beta, \cdots, \omega)$ is some permutations of $(1, 2, \cdots, n)$

Ex: $A = U$

The only nonzero term comes from the diagonal $\Rightarrow \det U = + u_{11} u_{22} \cdots u_{nn}$

(All other col. orderings picks at least one entry below the diagonal. Since all entries of U below the diagonal is zero, det $= 0$)

$\Rightarrow \det I = +(1)(1)\cdots(1) = 1$

(This formula satisfies property 1. You can chk property 2, 3 are also true)

Ex: Z is the identity matrix except col. 3

$$\det Z = \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{vmatrix} = +(1)(1)(c)(1)$$ (only nonzero term)

($\because$ If you pick a, b, or d, we used up col. 3. For row 3, we can only pick 0

$\Rightarrow$ row 3 = zero row $\Rightarrow$ det = 0)

Determinant by cofactors

Recall: For 3x3 matrix A

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) : C_{11}$$

$$+ a_{12}(a_{23}a_{31} - a_{21}a_{33}) : C_{12}$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31}) : C_{13}$$

((cofactors: 2x2 det comes from matrices in row 2 & 3)

or

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

(Still choose one entry from each col. and row when we split the det)

Let M_{ij} be a submatrix of size n-1 by crossing out 1st row & jth col of A

$$\Rightarrow \det A = a_{11}\det M_{11} - a_{12}\det M_{12} + a_{13}\det M_{13}$$

Note: $\begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{12} & & \\ & a_{21} & a_{23} \\ & a_{31} & a_{33} \end{vmatrix} = -a_{12}\det M_{12}$

(one row change)

(we need to watch signs)

$$\begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix} = -\begin{vmatrix} & a_{13} & \\ a_{21} & & a_{22} \\ a_{31} & & a_{32} \end{vmatrix} = (-1)^2 \begin{vmatrix} a_{13} & & \\ & a_{21} & a_{32} \\ & a_{31} & a_{32} \end{vmatrix}$$

(two row changes)

In general,

$$C_{ij} = (-1)^{i+j} \det M_{ij}$$

Cofactor expansion:

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

(Just another form of the "big formula")

Note: we can do the expansion for any row

The most general form (cofactor formula)

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

where $C_{ij} = (-1)^{i+j} \det M_{ij}$

Q: Can we do cofactor expansion down a col.?

Yes! $\because$ $\det A^T = \det A$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Important Note:

We can find det of order n recursively via the cofactor formula

Application: tridiagonal matrices

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$|A_1| = |1| = 1$$

$$|A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$|A_4| = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\,|A_3| - 1\,|A_2| = -1$$

In fact,

$$|A_n| = |A_{n-1}| - |A_{n-2}|$$

We have a sequence which repeats every 6 terms:

$|A_1| = 1$, $|A_2| = 0$, $|A_3| = -1$, $|A_4| = -1$

$|A_5| = 0$, $|A_6| = 1$, $|A_7| = 1$, $|A_8| = 0$