EE203001 SES #20

Determinants

The determinant is an important number associated with any square matrix

e.g., the matrix is invertible iff its determinant is nonzero

Notation:

$$\det(A) \text{ or } |A|$$

Properties

Start with properties $\rightarrow$ Big formula

(e.g., $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$)

Basic rules (1-3)

Rule (4-10) follows from 1-3

1. $\det I = 1$ for any $n \times n$ identity matrix I

2. The determinant changes sign when two rows are exchanged (sign reversal)

e.g., $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Note: we can find det P from rule 2

$\Rightarrow$ exchange rows of I to reach P

$\Rightarrow$ $\det P = +1$ (even number of row change)

$\det P = -1$ (odd " " ")

3. The det is a linear fcn of each row separately (other rows unchanged)

Chk 2x2:

(a) $$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

(b) $$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

(true for any row since by rule 2 we can put any row as row 1 then exchang it back. det won't change)

Note: $\det 2I \neq 2 \det I$

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 = 4, \quad \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2$$

(Just like area & volume)

From Rule 1-3, we can deduce many others (Rule 4-10)

4. If two rows of A are equal, then $\det A = 0$

Chk 2×2: $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 \quad (ab - ab = 0)$

Reason: By rule 2, we can exchange these two rows $\Rightarrow -D$ (if $\det A = D$)

But A stays the same when we exchange two identical rows $\Rightarrow D$

So we have $-D = D \Rightarrow D = 0$

5. Substracting a multiple of one row from another row leaves $\det A$ unchanged

Chk 2×2:

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Reason: (for 2×2)

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \overset{3(b)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$$

$$\overset{3(a)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{=0}$$

(Proof for higher dim is similar)

Conclusion: Det not changed by Elimination $\det A = \pm \det U$

(if row change)

6. A matrix with a row of zeros has

$\det A = 0$

Chk 2x2:

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0, \quad \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$$

Reason: 2x 2

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} \overset{5}{=} \begin{vmatrix} c & d \\ c & d \end{vmatrix} \overset{4}{=} 0$$

or

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} \overset{3(a)}{=} 0 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

7. If A is triangular then

$\det A = a_{11} a_{22} \cdots a_{nn}$ = product of diagonal entries

Chk 2x2:

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad, \quad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad$$

Reason: Do Gauss - Jordan elimination to eliminate entries in upper triangular for U (lower triangular for L)

$\Rightarrow$ We reach D with entries of diagonal of U, By Rule 5. det stays the same

& $\det D = a_{11} \cdots a_{nn} \det I$ by rule 1

Note: If $a_{ii} = 0$ for some i, Elimination produces a zero row $\overset{6}{\Rightarrow} \det A = 0$

8. $\det A = 0$ iff A is singular

Reason:

If A is singular, we can use elimination to get zero rows $\overset{6}{\Rightarrow} \det A = 0$

If A is not singular, elimination produces a full set of pivots $d_1, \cdots, d_n$ on U

$$\Rightarrow \det A \overset{5}{=} \underset{\underset{4}{\uparrow}}{\pm} \det U \overset{7}{=} \underset{\underset{9}{\uparrow}}{\pm} (d_1 d_2 \cdots d_n)$$

(possible row exchange)

Derive 2x2 formula

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \overset{8}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{vmatrix} = a\left(d - \frac{c}{a} b\right) = ad - bc$$

(In fact, we know how to derive det for any $n \times n$ invertible A

$\det A = \pm \det U = \pm (d_1 \cdots d_n)$. This is how MATLAB compute det!)

9. $\det(AB) = \det(A)\det(B)$

Chk 2x2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{vmatrix}$$

Reason: When $|B| \neq 0$, Let $D(A) = \dfrac{|AB|}{|B|}$

chk if $D(A)$ satisfies Rule 1-3

$\Rightarrow D(A) = |A|$

Rule 1: If $A = I$, $D(A) = \dfrac{|B|}{|B|} = 1$ (v)

Rule 2: When two rows of A are exchanged $\Rightarrow$ same two rows of AB are exchanged

$|AB|$ changes sign $\Rightarrow D(A) = \dfrac{|AB|}{|B|}$ changes sign

Rule 3: (a) When row 1 of A is multiplied by t so is row 1 of AB

$\det A'B = t \det AB$

$\Rightarrow D(A') = t\,D(A)$ (v)

(b) Add row 1 of A to row 1 of A' to get row 1 of A''

$\Rightarrow$ row 1 of $A''B$ = row 1 of AB + row 1 of $A'B$

3(b)

$\Rightarrow |A''B| = |AB| + |A'B|$

$\Rightarrow \dfrac{|A''B|}{|B|} = \dfrac{|AB|}{|B|} + \dfrac{|A'B|}{|B|}$

(trivial for $|B| = 0$) $\Rightarrow D(A'') = D(A) + D(A')$ (v)

Note: $AA^{-1} = I \Rightarrow \det(A)\det(A^{-1}) = 1$

$\Rightarrow \det(A^{-1}) = 1/\det(A)$

Note: $\det(A^2) = (\det A)^2$

10. $\det(A^T) = \det(A)$

Chk 2x2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

Reason:

If $|A| = 0 \Rightarrow A$ singular $\Rightarrow A^T$ singular

$\Rightarrow |A^T| = 0$

For invertible A, $PA = LU$

$\Rightarrow (PA)^T = (LU)^T \Rightarrow A^T P^T = U^T L^T$

Compare

$\det P \det A = \det L \det U$

&

$\det A^T \det P^T = \det U^T \det L^T$) 9

- $\det L = 1 = \det L^T$
  (both have 1's on diagonal)
- $\det U = d_1 \cdots d_n = \det U^T$
  (both U, U^T are triangular & have same diagonal entries)
- $\det P = \det P^T = \pm 1$

$$(P^T = P^{-1} \Rightarrow \det P^T = \det P^{-1} = 1/\det P)$$

$$\Rightarrow \det A = \det A^T$$

Note: By this property, every rules for rows can be applied to col.s, e.g., exchange two col.s $\Rightarrow$ det changes sign ...