EE 203001 SES # 19

Orthogonal matrices & Gram-Schmidt

Two goals in this SES:

Goal 1: See how orthogonal matrices make calculations of $\hat{x}$, $\underline{P}$, P easier

Goal 2: See how to obtain orthogonal matrices (Gram-Schmidt process)

Orthonormal vectors

$\boxed{\text{Def}}$ The vectors $\underline{q}_1, \underline{q}_2, \cdots \underline{q}_n$ are orthonormal if

$$\underline{q}_i^T \underline{q}_j = \begin{cases} 0 & \text{if } i \neq j \quad \text{(orthogonal)} \\ 1 & \text{if } i = j \quad \text{(unit vectors } \|\underline{q}_i\| = 1\text{)} \end{cases}$$

Note: orthonormal vectors are always indep.

Orthonormal matrices

Q is an orthonormal matrix if its col.s are orthonormal vectors
(Q can be rectangular)

$\boxed{\text{Fact}}$ For orthonormal matrix Q

$$Q^T Q = I$$

Reason:

$$Q^TQ = \begin{bmatrix} - \underline{q}_1^T - \\ \vdots \\ - \underline{q}_n^T - \end{bmatrix} \begin{bmatrix} | & & | \\ \underline{q}_1 & \cdots & \underline{q}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Note: If Q is square, we call it **orthogonal** matrix

In this case,

$$Q^TQ = I \Rightarrow Q^{-1} = Q^T \text{ (transpose = inverse)}$$

To repeat: $Q^TQ = I$ even when Q is rectangular (Q^T is only a left inverse)

For square Q, Q has full rank

$\Rightarrow Q^{-1}$ exist

$\Rightarrow Q^T$ is two-sided inverse $\Rightarrow Q^{-1} = Q^T$

$\Rightarrow Q^T$ is also the right inverse

$\Rightarrow$ we also have $QQ^T = I$

(Q also has orthonormal rows)

Important classes of matrix introduced so far: triangular, diagonal, permutation, symmetric, reduced row echelon, projection, and orthogonal matrices

Ex: Rotation matrix

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

orthogonal + unit vector

$(\cos^2\theta + \sin^2\theta = 1)$

$$Q^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

(rotate θ) (rotate $-\theta$ back)

$$= Q^{-1}$$

Ex: Permutation matrix (always orthogonal matrix)

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(both Q & Q^T are orthogonal matrices & $QQ^T = I$, $Q^TQ = I$)

Ex: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ NOT orthogonal matrix

orthogonal but NOT unit vector

normalize & get $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = Q_1$

Ex: Hadamard matrices

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} Q_1 & Q_1 \\ Q_1 & -Q_1 \end{bmatrix}$$

Ex: $\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$ (rectangular)

orthogonal but NOT unit vector

normalize & get

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix} \text{ (not square)}$$

we can add a 3rd col.

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix} \text{ (orthogonal matrix)}$$

$\boxed{\text{Fact}}$ If Q has orthonormal col.s ($Q^TQ = I$)

it leaves length unchanged, i.e.,

$$\|Q\underline{x}\| = \|\underline{x}\| \quad \forall\, \underline{x} \qquad \text{— ①}$$

Q also preserves dot products, i.e.,

$$(Q\underline{x})^T(Q\underline{y}) = \underline{x}^T\underline{y} \qquad \text{— ②}$$

Reason:

For ①, $\|Q\underline{x}\|^2 = (Q\underline{x})^T(Q\underline{x}) = \underline{x}^TQ^TQ\underline{x}$

$= \underline{x}^T\underline{x} = \|\underline{x}\|^2$

For ②, $(Q\underline{x})^T(Q\underline{y}) = \underline{x}^TQ^TQ\underline{y} = \underline{x}^T\underline{y}$

Projection using orthonormal bases : Q replaces A

$$A^T A \hat{\underline{x}} = A^T \underline{b}, \quad \underline{p} = A\hat{\underline{x}}, \quad P = A(A^TA)^{-1}A^T$$

$$\Downarrow$$

$$Q^T Q \hat{\underline{x}} = Q^T \underline{b}, \quad \underline{p} = Q\hat{\underline{x}}, \quad P = Q(Q^TQ)^{-1}Q^T$$

$$\Downarrow$$

$$\hat{\underline{x}} = Q^T \underline{b}, \quad \underline{p} = Q\hat{\underline{x}}, \quad P = QQ^T$$

$(\hat{x}_i = \underline{q}_i^T \underline{b})$

(projection is just a dot product)

$$\underline{p} = \begin{bmatrix} | & & | \\ \underline{q}_1 & \cdots & \underline{q}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \underline{q}_1^T \underline{b} \\ \vdots \\ \underline{q}_n^T \underline{b} \end{bmatrix}$$

$$= \underline{q}_1 (\underline{q}_1^T \underline{b}) + \cdots + \underline{q}_n (\underline{q}_n^T \underline{b})$$

Note : when Q is square, col.s of Q span the entire space

$$\hat{\underline{x}} = Q^T \underline{b}, \quad \underline{p} = Q\hat{\underline{x}}, \quad P = QQ^T$$

(least square sol.) $\Downarrow$

$$\underline{x} = Q^{-1} \underline{b}, \quad \underline{p} = QQ^{-1}\underline{b}, \quad P = I$$

(exact sol.) $= \underline{b}$

or

$$\underline{p} = \underline{b} = \underline{q}_1 (\underline{q}_1^T \underline{b}) + \cdots + \underline{q}_n (\underline{q}_n^T \underline{b})$$

(projection onto orthonormal basis & assemble it back)

(Foundation for Fourier series !)

Gram-Schmidt process

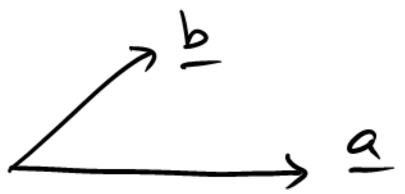
Elimination ⇒ make matrix triangular

Gram-Schmidt ⇒ make matrix orthonormal

Step 1: construct orthogonal vectors

Step 2: normalize to get orthonormal vectors

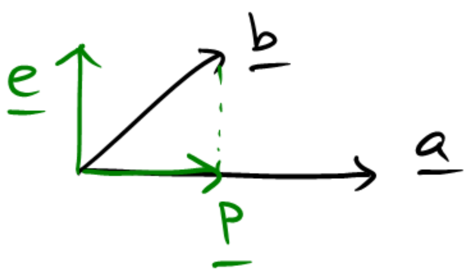
Start with two indep. vectors $\underline{a}, \underline{b}$



Find orthogonal vectors $\underline{A}, \underline{B}$ that span the same space

Q: How do we do that?

Set $\underline{A} = \underline{a}$



$\underline{e} \perp \underline{P} \Rightarrow \underline{e} \perp \underline{a}$

Set $\underline{B} = \underline{e}$

$$\Rightarrow \underline{B} = \underline{b} - \underline{P} = \underline{b} - \frac{\underline{A}^T \underline{b}}{\underline{A}^T \underline{A}} \underline{A}$$

(chk: $\underline{A}^T \underline{B} = \underline{A}^T \underline{b} - \frac{\underline{A}^T \underline{A}}{\underline{A}^T \underline{A}} \underline{A}^T \underline{b} = 0$)

(indeed orthogonal)

Q: What if we had 3rd indep. vector $\underline{c}$?

Substract components in the direction of $\underline{A}$ & $\underline{B}$ from $\underline{c}$

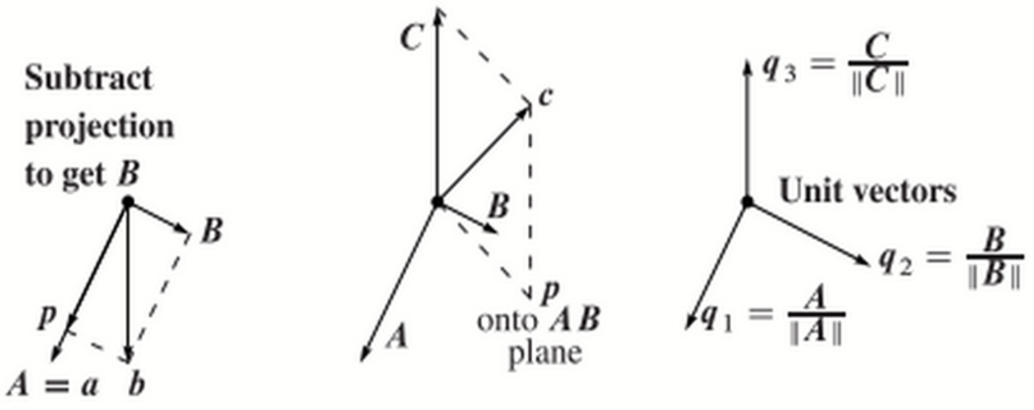


Figure 33: First project $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ and find the orthogonal $\boldsymbol{B}$ as $\boldsymbol{b}-\boldsymbol{p}$. Then project $\boldsymbol{c}$ onto the $\boldsymbol{AB}$ plane and find $\boldsymbol{C}$ as $\boldsymbol{c}-\boldsymbol{p}$. Divide by $\|\boldsymbol{A}\|, \|\boldsymbol{B}\|, \|\boldsymbol{C}\|$.

$$\underline{C} = \underline{c} - \frac{\underline{A}^T\underline{c}}{\underline{A}^T\underline{A}}\underline{A} - \frac{\underline{B}^T\underline{c}}{\underline{B}^T\underline{B}}\underline{B} \quad \left(\begin{array}{l}\underline{C} \perp \underline{A} \\ \underline{C} \perp \underline{B}\end{array}\right)$$

Step 2: $\underline{q}_1 = \frac{\underline{A}}{\|\underline{A}\|}, \quad \underline{q}_2 = \frac{\underline{B}}{\|\underline{B}\|}, \quad \underline{q}_3 = \frac{\underline{C}}{\|\underline{C}\|}$

(we can keep doing this to construct more orthonormal vectors)

(read Ex 5 in textbook p.235)

QR decomposition

Recall: When we studied Elimination, use Elimination matrices to represent the process $\Rightarrow$ leads to $A = LU$

$(EA = U \Rightarrow A = E^{-1}U = LU)$

A similar eqn $A = QR$ relates A to Q of the Gram-Schmidt process

($Q^TA = Q^TQR = R \Rightarrow R = Q^TA$)

$$\Rightarrow \underset{A}{\begin{bmatrix} | & | & | \\ \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \\ | & | & | \end{bmatrix}} = \underset{Q}{\begin{bmatrix} | & | & | \\ \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \\ | & | & | \end{bmatrix}} \underset{R}{\begin{bmatrix} \underline{q}_1^T\underline{a}_1 & \underline{q}_1^T\underline{a}_2 & \underline{q}_1^T\underline{a}_3 \\ & \underline{q}_2^T\underline{a}_2 & \underline{q}_2^T\underline{a}_3 \\ & & \underline{q}_3^T\underline{a}_3 \end{bmatrix}}$$

(R is upper triangular since later $\underline{q}$'s are chosen to be orthogonal to earlier $\underline{a}$'s e.g. $\underline{q}_2^T\underline{a}_1 = 0$, $\underline{q}_3^T\underline{a}_1 = 0$ ...)

This is Gram-Schmidt in a nutshell:

- $\underline{a}_1$ & $\underline{q}_1$ are along a single line
- $\underline{a}_1, \underline{a}_2$ & $\underline{q}_1, \underline{q}_2$ on the same plane ($\underline{a}_1, \underline{a}_2$ are comb. of $\underline{q}_1, \underline{q}_2$)
- $\underline{a}_1, \underline{a}_2, \underline{a}_3$ & $\underline{q}_1, \underline{q}_2, \underline{q}_3$ in one subspace (dim = 3) ($\underline{a}_1, \underline{a}_2, \underline{a}_3$ are comb. of $\underline{q}_1, \underline{q}_2, \underline{q}_3$)

In general, $\underline{a}_1, \cdots, \underline{a}_k$ are comb. of $\underline{q}_1, \cdots, \underline{q}_k$ only $\Rightarrow R$ is upper triangular

Solving least squares problem

$$A\underline{x} = \underline{b} \quad (\text{no sol.})$$

$$A^T A \underline{\hat{x}} = A^T \underline{b} \quad (\text{using } QR)$$

$$(QR)^T QR \underline{\hat{x}} = R^T Q^T \underline{b}$$

$$\Rightarrow R^T R \underline{\hat{x}} = R^T Q^T \underline{b} \quad \text{or} \quad R\underline{\hat{x}} = Q^T \underline{b}$$

$$\text{or} \quad \underline{\hat{x}} = R^{-1} Q^T \underline{b}$$

(can be easily solved using back sub.)