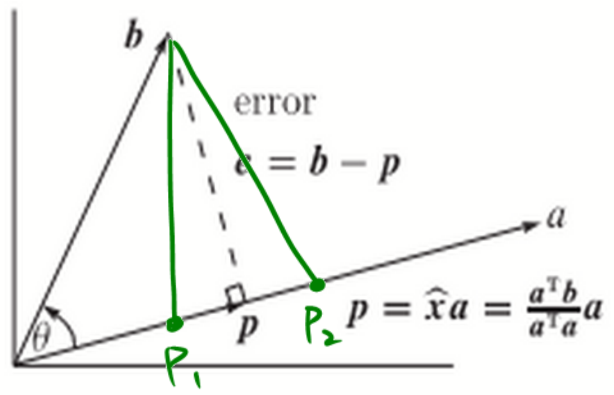
Projections

Projection onto a line



Q: How do we find a point $\underline{P}$ on the line (determined by vector $\underline{a}$) that is closest to $\underline{b}$?

$\underline{P}$: intersection of a line through $\underline{b}$ that is orthogonal to $\underline{a}$

($\underline{P_1}$, $\underline{P_2}$ have longer distance)

More precisely

Think of $\underline{P}$ as an approx. of $\underline{b}$

then $\underline{e} = \underline{b} - \underline{P}$ is the error vector

Since $\underline{P}$ is along the line of $\underline{a}$

$\Rightarrow \underline{P} = \hat{x}\,\underline{a}$ for some $\hat{x}$

Also, $\underline{a} \perp \underline{e}$

$$\Rightarrow \underline{a}^T(\underline{b} - \underline{P}) = \underline{a}^T(\underline{b} - \hat{x}\underline{a}) = 0$$

$$\Rightarrow \underline{a}^T\underline{a}\,\hat{x} = \underline{a}^T\underline{b} \Rightarrow \hat{x} = \frac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}}$$

Now, we have

$$\underline{p} = \hat{x}\,\underline{a} = \underline{a}\,\hat{x} = \underline{a}\,\frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}}$$

(doubling $\underline{b}$ doubles $\underline{p}$, doubling $\underline{a}$ does NOT affect $\underline{p}$)

<u>Projection matrix</u> ($\underline{p} = P\underline{b}$)

$$\underline{p} = \underline{a}\,\frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} = \underbrace{\frac{\underline{a}\,\underline{a}^T}{\underline{a}^T \underline{a}}}_{P = \frac{\underline{a}\,\underline{a}^T}{\underline{a}^T \underline{a}}}\,\underline{b}$$

(For 3D)

$\underline{a}\,\underline{a}^T \leftarrow$ a 3×3 rank-one matrix

$\underline{a}^T\underline{a} \leftarrow$ a number

(procedure: Find $\hat{x} \to \underline{p} \to P$)

Special case I: If $\underline{b} = \underline{a}$, $\hat{x} = 1$

$\Rightarrow P\underline{a} = \underline{a}$ (proj. of $\underline{a}$ onto $\underline{a}$ is itself)

Special case II: If $\underline{b} \perp \underline{a}$, $\underline{a}^T\underline{b} = 0$

$\Rightarrow \underline{p} = \underline{0}$

Note 1: col. space of P is spanned by $\underline{a}$ ($\because$ for any $\underline{b}$, $P\underline{b}$ lies on the line determined by $\underline{a}$)

Note 2: rank$(P) = 1$

Note 3: P is symmetric

$$\left(P^T = \frac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}} = \frac{1}{\underline{a}^T\underline{a}}\left(\underline{a}\,\underline{a}^T\right)^T = \frac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}} = P \right)$$

Note 4: $P^2 = P$

($P^2 \underline{b} = P\underline{b}$ or $P(P\underline{b}) = P\underline{b}$

$\because$ proj. of a vector already on $\underline{a}$ is

itself)

Note 5 : $I-P$ is also a projection

($(I-P)\underline{b} = \underline{b} - \underline{p} = \underline{e}$ in the left nullspace

of $\underline{a}$ $\because$ $\underline{a}^T\underline{e} = 0$)

(P : project onto one subspace

$I-P$: " " the perpendicular subspace)

<u>Q : Why project ?</u>

$A\underline{x} = \underline{b}$ may have no sol.

always in col. space of A

unlikely that $\underline{b} \in C(A)$

If not, project $\underline{b}$ onto $\underline{p} \in C(A)$

then solve $A\hat{x} = \underline{p}$

<u>Projection onto a subspace</u>

<u>Projection onto a plane</u> (in $\mathbb{R}^3$)

If $\underline{a}_1, \underline{a}_2$ are basis of a plane

$\Rightarrow$ the plane is $C(A)$ of $A = [\underline{a}_1 \; \underline{a}_2]$

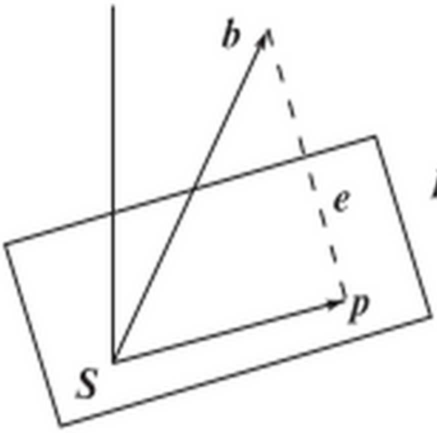
In general, for a subspace $S \subseteq \mathbb{R}^m$ with n indep. basis $\underline{a_1}, \cdots, \underline{a_n}$

$\Rightarrow$ subspace is $C(A)$ of $\underset{m \times n}{A} = [\underline{a_1}, \cdots, \underline{a_n}]$

Problem: Find $\underline{P}$ in S closest to $\underline{b}$

since $\underline{P} \in C(A)$, $\underline{P} = A\hat{\underline{x}} = \hat{x}_1 \underline{a_1} + \cdots + \hat{x}_n \underline{a_n}$

(want to find $\hat{x}_i$)



$p = A\hat{x}$
$= A(A^T A)^{-1} A^T b$
$= Pb$

$\underline{P}$ closest to $\underline{b}$

$\Rightarrow \underline{e} = \underline{b} - \underline{P} \perp S$

or $\underline{b} - A\hat{\underline{x}} \perp S$

$\Rightarrow \underline{e} = \underline{b} - A\hat{\underline{x}}$ perpendicular with $\underline{a_1}, \cdots, \underline{a_n}$

$$\Rightarrow \begin{aligned} \underline{a_1}^T(\underline{b} - A\hat{\underline{x}}) &= 0 \\ \underline{a_2}^T(\underline{b} - A\hat{\underline{x}}) &= 0 \\ &\vdots \\ \underline{a_n}^T(\underline{b} - A\hat{\underline{x}}) &= 0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} - \underline{a_1}^T - \\ \vdots \\ - \underline{a_n}^T - \end{bmatrix} \begin{bmatrix} \underline{b} - A\hat{\underline{x}} \end{bmatrix} = \begin{bmatrix} \underline{0} \end{bmatrix}$$

$\Rightarrow A^T(\underbrace{\underline{b} - A\hat{\underline{x}}}_{= \underline{e}}) = \underline{0} \Rightarrow A^T A \hat{\underline{x}} = A^T \underline{b}$

(in left nullspace of A)

Find $\hat{\underline{x}}$

$$\hat{\underline{x}} = (A^T A)^{-1} A^T \underline{b}$$

(Q: Is A^TA invertible ?

Yes, if n col.s of A are lin. indep.)

(will prove this later)

Find $\underline{p}$

$$\underline{p} = A\hat{\underline{x}} = \underbrace{A(A^TA)^{-1}A^T}\underline{b}$$

projection matrix $P = A(A^TA)^{-1}A^T$

(Find $\hat{\underline{x}} \to \underline{p} \to P$)

Alternative derivation

1. Our subspace is $C(A)$
2. error vector $\underline{e} = \underline{b} - A\hat{\underline{x}} \perp C(A)$
3. So $\underline{e}$ in left nullspace of A

   ($C(A)$ & $N(A^T)$ are orthogonal complements)

$$\Rightarrow A^T\underline{e} = A^T(\underline{b} - A\hat{\underline{x}}) = \underline{0}$$

($\underline{b}$ splitted into $\underline{p}$ & $\underline{e}$)
$(\in C(A))$ $(\in N(A^T))$

Special cases

1. $\underline{b} \perp C(A)$: $\underline{b} \in N(A^T)$ & $P\underline{b} = \underline{0}$
2. $\underline{b} \in C(A)$: $A\underline{x} = \underline{b}$ for some $\underline{x}$
   & $P\underline{b} = \underline{b}$

Q: Can we further simplify $P = A(A^TA)^{-1}A^T$?

$$P = A(A^TA)^{-1}A^T = A(A^{-1}(A^T)^{-1})A^T$$
$$= (AA^{-1})((A^T)^{-1}A^T)$$
$$= I \ ?$$

Wrong! A is rectangular $\Rightarrow$ A has No inverse matrix!

$\boxed{\text{Fact}}$ $P = P^T$, $P^2 = P$ (still true for general n)

distance from $\underline{b}$ to the subspace $= \|\underline{e}\|$

$\boxed{\text{Fact}}$ A^TA is invertible iff A has lin. indep. col.s

pf: First, we want to show that A^TA & A have same nullspace

If $\underline{x}$ is in $N(A)$, then $A\underline{x} = \underline{0}$

$$\Rightarrow A^TA\underline{x} = A^T(\underline{0}) = \underline{0}$$
$$\Rightarrow \underline{x} \text{ in } N(A^TA)$$

If $\underline{x}$ in $N(A^TA)$, then $A^TA\underline{x} = \underline{0}$

$$\Rightarrow \underline{x}^TA^TA\underline{x} = \underline{x}^T\underline{0} = 0$$
$$\Rightarrow \|A\underline{x}\|^2 = 0 \Rightarrow A\underline{x} = \underline{0} \Rightarrow \underline{x} \in N(A)$$

So A & A^TA have same nullspace

Now, if A has indep. col.s

then $\text{rank}(A) = n \Rightarrow N(A) = \{\underline{0}\}$

$\Rightarrow N(A^TA) = \{\underline{0}\} \Rightarrow A^TA$ is invertible

If A^TA invertible, then A^TA has indep. col.s $\Rightarrow N(A^TA) = \{\underline{0}\} \Rightarrow N(A) = \{\underline{0}\}$

$\Rightarrow A$ has indep. col.s

Ex 3: (on p. 211, textbook)

If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$, find $\hat{x}$, p & P

Normal eqn:

$$A^TA\underline{\hat{x}} = A^T\underline{b}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$\underline{p} = A\underline{\hat{x}} = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$\underline{e} = \underline{b} - \underline{p} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$ (indeed $\perp$ both cols of A)

To find $\underline{p}$ for every $\underline{b}$, we need P

$$P = A(A^TA)^{-1}A^T$$

$$(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \Rightarrow P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

(chk: $P\underline{b} = \underline{p}$ & $P^2 = P$, $P^T = P$)