EE 203001 SBS # 16

Orthogonality of the Four subspaces

Looking ahead (part I) (part II)

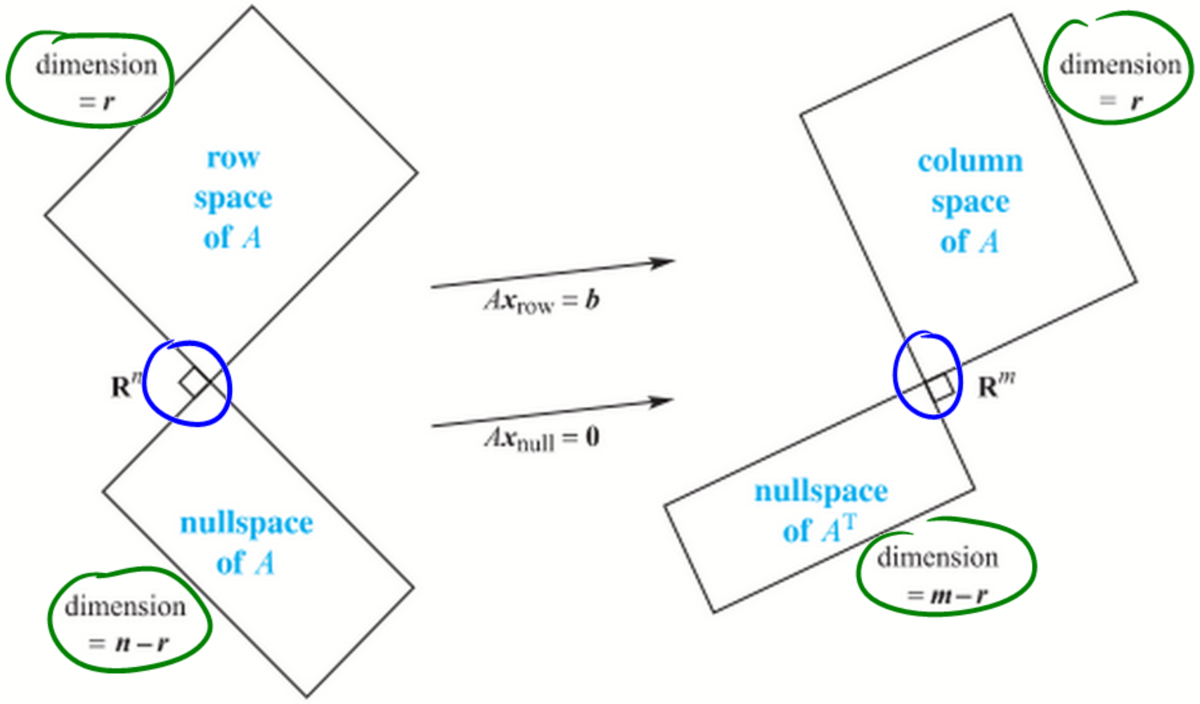
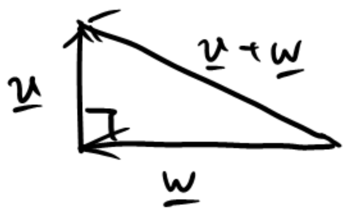


Figure 24: Two pairs of orthogonal subspaces. The dimensions add to n and add to m. **This is an important picture**—one pair of subspaces is in $\mathbf{R}^n$ and one pair is in $\mathbf{R}^m$.

Orthogonal vectors



Two vectors are orthogonal (= perpendicular) if $\underline{v}^T \underline{w} = 0$

or $\|\underline{v}+\underline{w}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2$

$$\left((\underline{v}+\underline{w})^T(\underline{v}+\underline{w}) = \underline{v}^T\underline{v} + \underline{w}^T\underline{w} + \underline{w}^T\underline{v} + \underline{v}^T\underline{w} \right.$$

$$\left. = \|\underline{v}\|^2 + \|\underline{w}\|^2 \Rightarrow \underline{w}^T\underline{v} = \underline{v}^T\underline{w} = 0 \right)$$

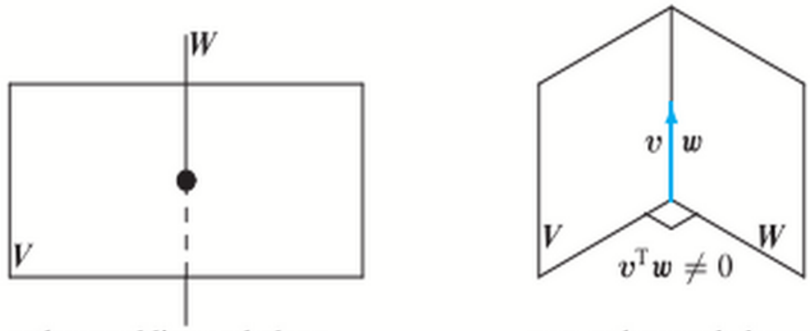
Note: All vectors are orthogonal to zero vector

Orthogonal subspaces

$\boxed{\text{Def}}$ Subspace S is orthogonal to subspace T if every vector in S is orthogonal to every vectors in T

$(\underline{v}^T \underline{w} = 0 \quad \forall\ \underline{v} \in S,\ \forall\ \underline{w} \in T)$

Ex:



orthogonal line and plane

non-orthogonal planes

Figure 23: Orthogonality is impossible when dim V + dim W > dimension of whole space.

(If $S \cap T$ contains any vector (except $\underline{0}$)

$\Rightarrow$ S & T cannot be orthogonal)

($\underline{0}$ is orthogonal to itself $\underline{0}^T \underline{0} = 0$)

Nullspace is orthogonal to row space

$N(A)$ & $C(A^T)$ are orthogonal subspaces of R^n

Why? $\forall\ \underline{x} \in N(A)$, $A\underline{x} = \underline{0}$

$$A\underline{x} = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } m \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \cdot \underline{x} = 0 \\ \leftarrow \text{row } 2 \cdot \underline{x} = 0 \\ \vdots \\ \leftarrow \text{row } m \cdot \underline{x} = 0 \end{matrix}$$

$\Rightarrow$ $\underline{x}$ is orthogonal to every row of A

so it's also orthogonal to all comb. of rows of A $\Rightarrow N(A) \perp C(A^T)$

Left Nullspace is orthogonal to col. space

$N(A^T) \perp C(A)$: both orthogonal subspaces of $\mathbb{R}^m$

Reason: $\forall \underline{y} \in N(A^T)$, $A^T \underline{y} = \underline{0}$

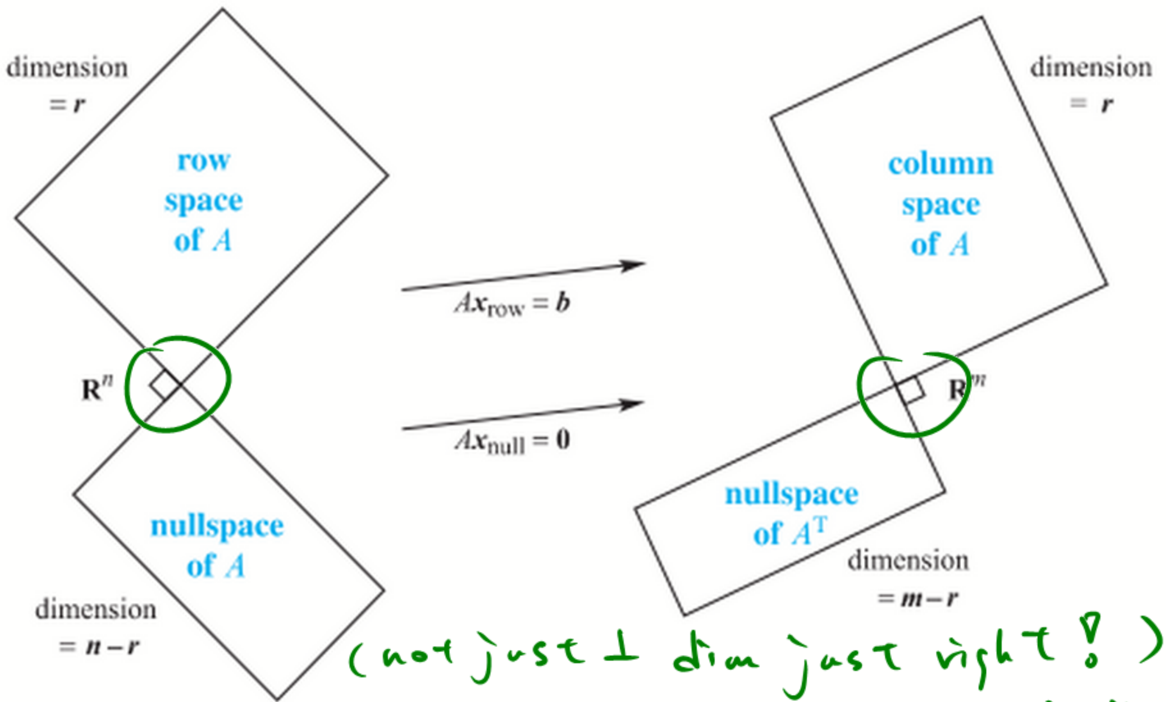
$$A^T \underline{y} = \begin{bmatrix} \text{col } 1^T \\ \text{col } 2^T \\ \vdots \\ \text{col } n^T \end{bmatrix} \underline{y} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow$ $\underline{y}$ is orthogonal to every col. of A

$\Rightarrow$ " " " " all comb. of col.s of A $\Rightarrow$ $N(A^T) \perp C(A)$

Orthogonal complements (V perp)

$\boxed{\text{Def}}$ The orthogonal complement $V^\perp$ of subspace V contains every vector perpendicular to V



(not just $\perp$ dim just right!)
$(r + (n-r) = n \;,\; r + (m-r) = m)$

Figure 24: Two pairs of orthogonal subspaces. The dimensions add to n and add to m. **This is an important picture**—one pair of subspaces is in $\mathbf{R}^n$ and one pair is in $\mathbf{R}^m$.

Fundamental Thm of Linear Algebra (part II)

(1) $N(A)$ is the orthogonal complement of $C(A^T)$ (in $\mathbb{R}^n$)

(2) $N(A^T)$ .. .. of $C(A)$ (in $\mathbb{R}^m$)

Reason for (1):

$\forall \underline{x}$ orthogonal to rows of A, $A\underline{x} = \underline{0}$

$\Rightarrow \underline{x} \in N(A) \Rightarrow N(A) = C(A^T)^\perp$

(reverse is also true, i.e., $C(A^T) = N(A)^\perp$

pf by contradiction: if $\exists \underline{v}$ orthogonal to $N(A)$ but not in $C(A^T)$, we can add $\underline{v}$ as a new row of matrix: $A' = \begin{bmatrix} A \\ \underline{v}^T \end{bmatrix}$ without changing $N(A)$ (If $A\underline{x} = \underline{0}$, then $A'\underline{x} = \underline{0}$ since $v^T x = 0$)

then $\dim C(A'^T) = \dim C(A^T) + 1 = r + 1$

but $\dim N(A') = \dim N(A) = n - r$

$\Rightarrow (n-r) + (r+1) = n+1 \neq n$ (contradiction!)

(Reason for (2) follows by changing A to A^T)

Ex: $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$

$\dim C(A^T) = 1 \Rightarrow \dim N(A) = 3 - 1 = 2$

(basis: $(1,2,5)$) (basis: two special sols)

By orthogonal complement, $N(A)$ is the plane perpendicular to $(1,2,5)$

Row space & Nullspace components

Since $C(A^T)$ & $N(A)$ are orthogonal complements

$$(C(A^T) = N(A)^\perp \ \& \ N(A) = C(A^T)^\perp)$$

every $\underline{x} \in \mathbb{R}^n$ can be splitted into

$$\underline{x} = \underbrace{\underline{x}_r}_{\text{(row space component)}} + \underbrace{\underline{x}_n}_{\text{(nullspace component)}}$$

(will prove this later)

Multiplying by A

Note 1: $A\underline{x}_n = \underline{0}$ (nullspace component goes to zero)

Note 2: $A\underline{x}_r = A\underline{x} = \underline{b}$

(row space component goes to $C(A)$)

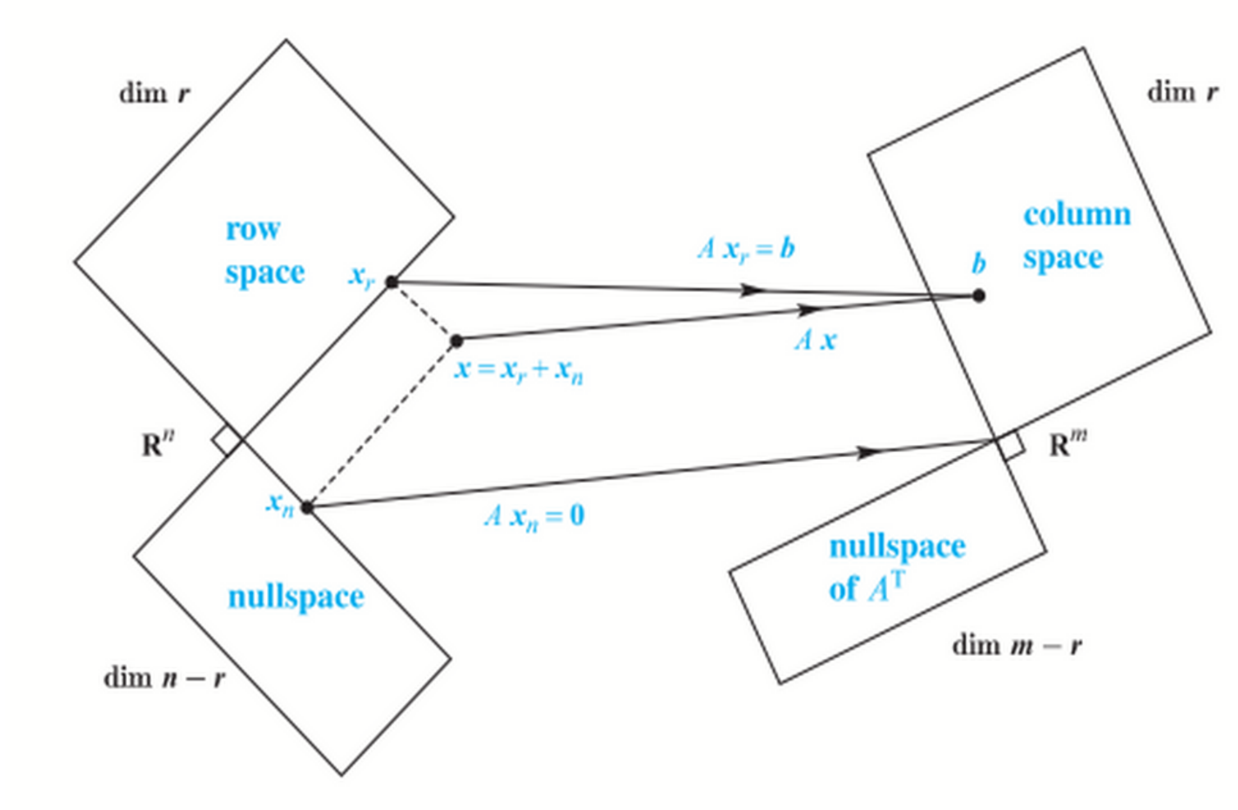


Figure 25: This update of Figure 24 shows the true action of A on $x = x_r + x_n$. Row space vector x_r to column space, nullspace vector x_n to zero.

Note 3: Every vector $\underline{b} \in C(A)$ comes from a unique vector $\underline{x}_r \in C(A^T)$

(pf: If $\exists\ \underline{x}_r, \underline{x}_r' \in C(A^T)$ s.t.

$$A\underline{x}_r = A\underline{x}_r' \Rightarrow A\underline{x}_r - A\underline{x}_r' = \underline{0}$$

$$\Rightarrow A(\underline{x}_r - \underline{x}_r') = \underline{0} \Rightarrow \underline{x}_r - \underline{x}_r' \in N(A)$$

since $\underline{x}_r, \underline{x}_r' \in C(A^T) \Rightarrow \underline{x}_r - \underline{x}_r' \in C(A^T)$

$\Rightarrow \underline{x}_r - \underline{x}_r' = \underline{0}$ since $N(A) \perp C(A^T)$)

This implies that there is a $r \times r$ invertible matrix hiding inside A

(From $C(A^T) \to C(A)$, A is invertible & pseudoinverse will invert it in sec. 7.3)

Ex:

$$A = \begin{bmatrix} \boxed{\begin{matrix} 3 & 0 \\ 0 & 5 \end{matrix}} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \end{matrix} \end{bmatrix} \quad \text{invertible}$$

$$B = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} & \begin{matrix} 2 \\ 2 \end{matrix} & \boxed{\begin{matrix} 3 \\ 4 \end{matrix}} & \begin{matrix} 4 & 5 \\ 5 & 6 \end{matrix} \\ 1 & 2 & 4 & \begin{matrix} 5 & 6 \end{matrix} \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ invertible}$$

Combining Bases from subspaces

Recall: Basis = indep. + span the space

But when the count is right, we only need one of them, i.e.,

- Any n indep. vectors in R^n must span $R^n \Rightarrow$ they are basis

- Any n vectors that span R^n must be indep. $\Rightarrow$ they are basis

Equivalent statements: $(A_{n\times n})$

- If n col.s of A are indep. they span R^n $\Rightarrow A\underline{x}=\underline{b}$ solvable $\forall\ \underline{b}$
- If n col.s of A span R^n, they are indep. $\Rightarrow A\underline{x}=\underline{b}$ has only one sol.

(pf: If n col.s indep. then no free var.s $\Rightarrow$ sol. $\underline{x}$ unique & n pivots

$\Rightarrow$ back sub. solves $A\underline{x}=\underline{b}$

$\Rightarrow$ sol. exists

If n col.s span R^n, $A\underline{x}=\underline{b}$ is solvable $\forall\ \underline{b}$ (sol. exists)

$\Rightarrow$ n pivots $\Rightarrow$ no free var.s

$\Rightarrow$ sol. unique)

Combining bases from $C(A^T)$ & $N(A)$

We have r basis from $C(A^T)$ in R^n

(n-r) " " $N(A)$ " R^n

Combined together

A total of $r+(n-r)=n$ indep. vectors in R^n, they span R^n

(If $\underbrace{a_1 \underline{v}_1 + \cdots + a_r \underline{v}_r}_{\underline{x}_r} + \underbrace{a_{r+1} \underline{v}_{r+1} + \cdots + a_n \underline{v}_n}_{\underline{x}_n} = \underline{0}$

$\Rightarrow \underline{x}_r = -\underline{x}_n \Rightarrow \underline{x}_r, \underline{x}_n$ in both

$C(A^T)$ & $N(A)$ but $C(A^T) \perp N(A)$

$\Rightarrow \underline{x}_r = \underline{x}_n = \underline{0}$

Since $\underline{v}_1 \cdots \underline{v}_r$ are basis of $C(A^T)$

$\underline{v}_{r+1} \cdots \underline{v}_n$ " " " $N(A)$

$\Rightarrow a_1 = a_2 = \cdots a_r = a_{r+1} = \cdots = a_n = 0$

n vectors are indep.)

So for every $\underline{x}$ in $\mathbb{R}^n$, we have

$$\underline{x} = \underline{x}_r + \underline{x}_n$$