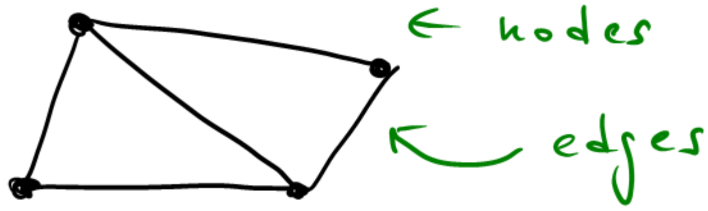
Small world graphs

$G = \{\text{nodes}, \text{edges}\}$

= collection of nodes joined by edges



Social network

Each node is a person, two nodes are connected by edges if they are friends

Q: What is the farthest distance between two people in the graph?

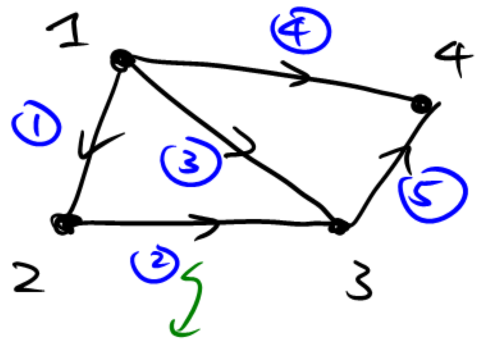
Six degrees of separation

⇒ It's a small world!

Other example

WWW: nodes are websites
edges are links

Electrical network



$n=4$ nodes

$m=5$ edges

(direction of currents) (Directed graph)

Incidence matrix

One col. for each node, one row for each edge

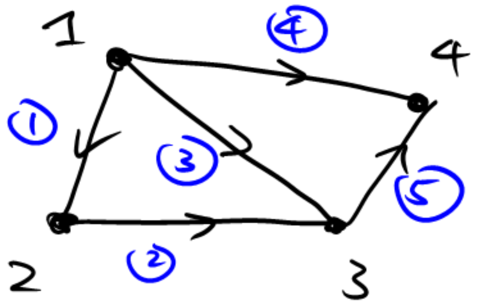
If edge runs from node 1 $\to$ node 2
(−1 in col 1) (+1 in col 2)

$$A = \begin{array}{c c}
\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{l} \text{edge } 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \\
\begin{array}{cccc} \text{Node } 1 & 2 & 3 & 4 \end{array} &
\end{array}$$

(Incidence matrix A if large is sparse in general $\Rightarrow$ most entries are zero)

(Each row only has 2 nonzero entries)

Loops:



$n = 4$ nodes

$m = 5$ edges

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} \left.\begin{matrix}\text{edge } 1 \\ 2 \\ 3\end{matrix}\right\} \text{lin. dependent} \\ 4 \\ 5 \end{matrix}$$

Node 1 2 3 4

(row 3 = row1 + row 2)

Null space of A

$\underline{x} = (x_1, x_2, x_3, x_4)$: potentials at nodes

$A\underline{x} = \underline{0}$

$$\Rightarrow A\underline{x} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{x} = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(diff. of potentials)

$$\Rightarrow \dim N(A) = 1 \text{ with basis } \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(Nothing will move if all potentials are the same) (or potential diff. = 0)

(But potentials can only be determined up to a constant)

(If we ground node 4, $x_4 = 0$ $\Rightarrow x_1 = x_2 = x_3 = 0$)

Q: What is rank(A)?

$$\text{rank}(A) + \dim N(A) = n = 4$$

$$\Rightarrow \text{rank}(A) = 4 - 1 = 3$$

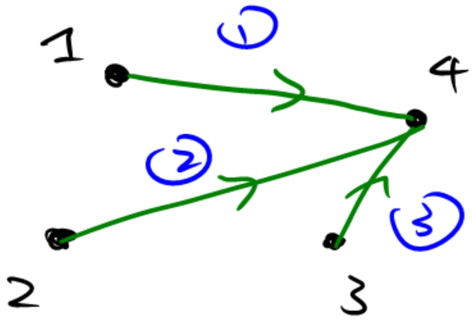
(We can also see this via Elimination)

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for C(A)

pivot col.

(Top 3 rows of R are indep. $\Rightarrow$ the graph it forms has no loops $\Rightarrow$ It's a tree!)



(This is a tree with no loops)

Left nullspace $N(A^T)$

$A^T \underline{y} = \underline{0}$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\dim N(A^T) = m - r = 5 - 3 = 2$

($\underline{y} = (y_1, y_2, y_3, y_4, y_5)$ are currents

& $A^T \underline{y} = 0$ is Kirchoff's current law)

(will come back to this later)

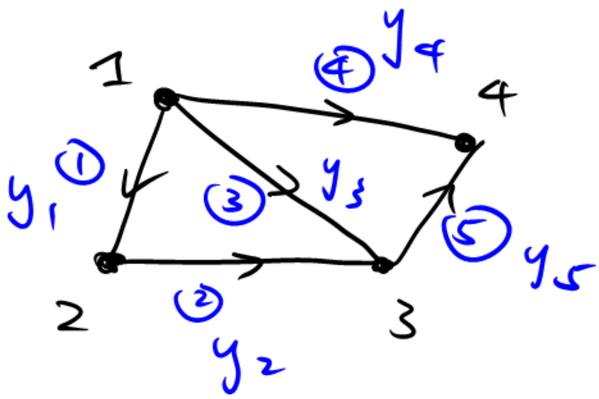
Basis :

node 1: $-y_1 - y_3 - y_4 = 0$

node 2: $y_1 - y_2 = 0$

node 3: $y_2 + y_3 - y_5 = 0$

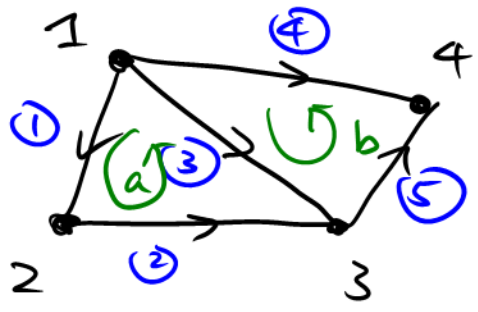
node 4: $y_4 + y_5 = 0$



(currents in = currents out)

($-y_1$: current out, $+y_1$: current in)

Basis by inspection:



current in loops

loop: $\underline{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

(loop a) (loop b)

(same as # of loops)

↓

(Dim $N(A^T) = 2$ so only need these two vectors for a basis)

(Outer loop also gives a special sol. (1, 1, 0, −1, 1))

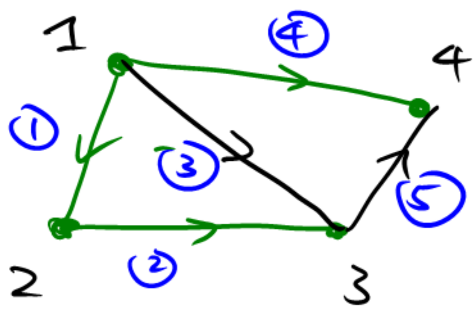
<u>Row space</u> $C(A^T)$

$$A^T = \begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

not a pivot col. since node 1, 2, 3 forms a loop

pivot col.s

$\text{rank}(A) = 3 \Rightarrow \dim C(A^T) = 3$



(lin indep. edges 1, 2, 4 form a tree)

Complete picture

$\underline{x} = (x_1, x_2, x_3, x_4)$
potentials at nodes

$\downarrow A\underline{x} = \underline{e}$

$x_2 - x_1$, etc.
potential differences

(conductance matrix)

$\underline{y} = -C\underline{e}$
$\longrightarrow$
Ohm's Law

$A^T\underline{y} = \underline{0}$
Kirchoff's current law

$\uparrow A^T\underline{y}$

y_1, y_2, y_3, y_4, y_5
currents on edges

Euler's formula

$\dim N(A^T) = m - r$

$\#\text{loops} = \#\text{edges} - (\#\text{nodes} - 1)$

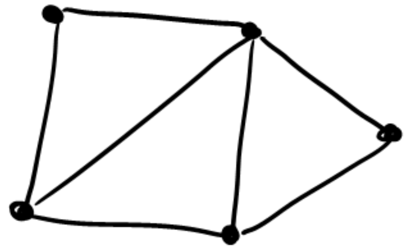
$(\text{rank} = n-1)$ $(\dim N(A)$ always $= 1)$

$\Rightarrow \#\text{nodes} - \#\text{edges} + \#\text{loops} = 1$

(small loops)

(True for any connected graph)

Ex:



$5 - 7 + 3 = 1$

One more thing

Still need a outside source to drive the circuit

Current source $\underline{f}$

$\underline{x} = (x_1, x_2, x_3, x_4)$ potentials at nodes

$\downarrow A\underline{x} = \underline{e}$

$x_2 - x_1$, etc. potential differences

(conductance matrix)

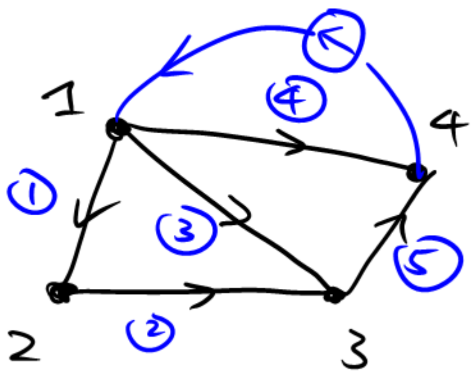
$\underline{y} = -C\underline{e}$

$\longrightarrow$ Ohm's Law

y_1, y_2, y_3, y_4, y_5 currents on edges

$\uparrow A^T\underline{y}$

$A^T\underline{y} + \underline{f} = \underline{0}$ Kirchoff's current law



(outside current source $\underline{f} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$)

Combining all 3 eqns

$$\underbrace{A^T C A}\,\underline{x} = \underline{f}$$

Symmetric matrix

Ex 1 in textbook (p. 427)

Ex 1: All conductances are $c = 1$

so $C = I$

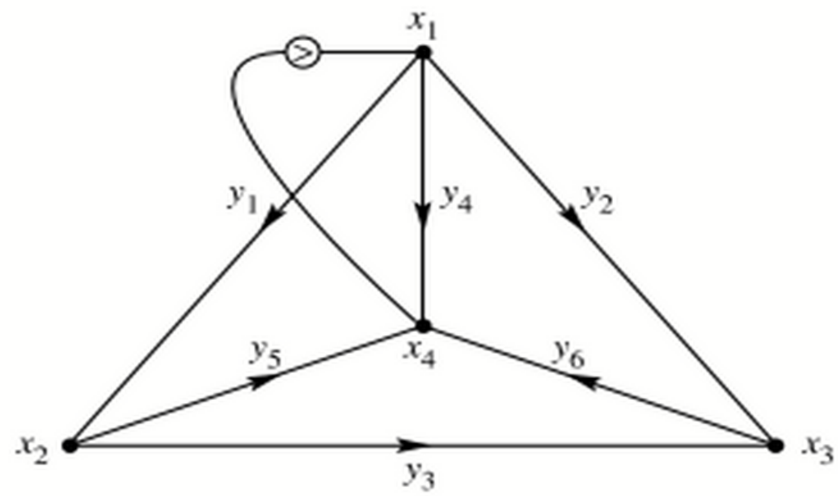


Figure 56: The currents in a network with a source S into node 1.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow A^T C A = A^T A$$

$$= \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Ground node 4 $\Rightarrow x_4 = 0$

$\Rightarrow$ remove col 4 & row 4 from A^TCA

Solve

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/4 \\ 5/4 \end{bmatrix}$$

By Ohm's law $y = -CA\underline{x}$ $(x_4 = 0)$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = - \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5/2 \\ 5/4 \\ 5/4 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 5/4 \\ 0 \\ 5/2 \\ 5/4 \\ 5/4 \end{bmatrix}$$