EE203001 SES #9

Space of vectors

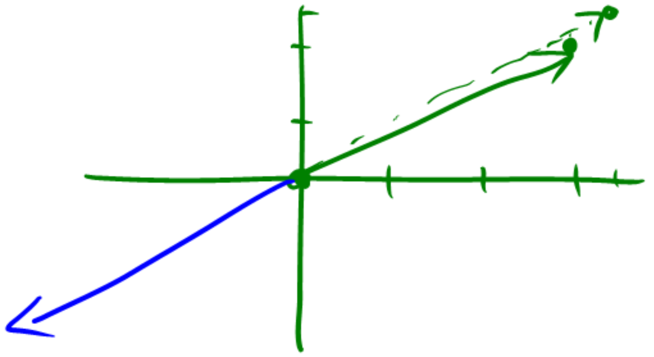
$\boxed{\text{Def}}$ A vector space is a collection of vectors which is closed under lin. combinations

(For any $\underline{v}$ & $\underline{w}$ in the space, $c\underline{v}+d\underline{w}$ is also in the space, c, d are any real numbers)

Ex:

R^2: All 2-D real vectors

($\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}$



closed under ADD

" " MUL)

(R^2: x-y plane)

More generally,

$\boxed{\text{Def}}$ The space R^n consists of all col. vectors $\underline{v}$ with n real components

(for complex components, we have C^n)

ADD & MUL need to follow 8 rules:

(1) $\underline{x} + \underline{y} = \underline{y} + \underline{x}$

(2) $\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z}$

(3) $\exists$ a unique zero vector $\underline{0}$ s.t.

$$\underline{x} + \underline{0} = \underline{x} \quad \forall\ \underline{x}$$

(4) For each $\underline{x}$, $\exists$ unique $-\underline{x}$ s.t.

$$\underline{x} + (-\underline{x}) = \underline{0}$$

(5) $1 \cdot \underline{x} = \underline{x}$

(6) $(c_1 c_2)\underline{x} = c_1 (c_2 \underline{x})$

(7) $c(\underline{x} + \underline{y}) = c\underline{x} + c\underline{y}$

(8) $(c_1 + c_2)\underline{x} = c_1 \underline{x} + c_2 \underline{x}$

EX: If we define $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_2 \\ x_2 + y_1 \end{bmatrix}$

with usual mul. $c\underline{x} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$

Do we still satisfy the 8 rules?

No! $\underline{x} + \underline{y} \neq \underline{y} + \underline{x}$

$$\underline{x} + (\underline{y} + \underline{z}) \neq (\underline{x} + \underline{y}) + \underline{z}$$

$$(c_1 + c_2)\underline{x} \neq c_1 \underline{x} + c_2 \underline{x}$$

Other examples of vector spaces:

M: vector space of all real 2x2 matrices

F: " " " " " fcns $f(x)$

Z: " " that consists only of a zero vector

(We can ADD, MUL, still in vector space)

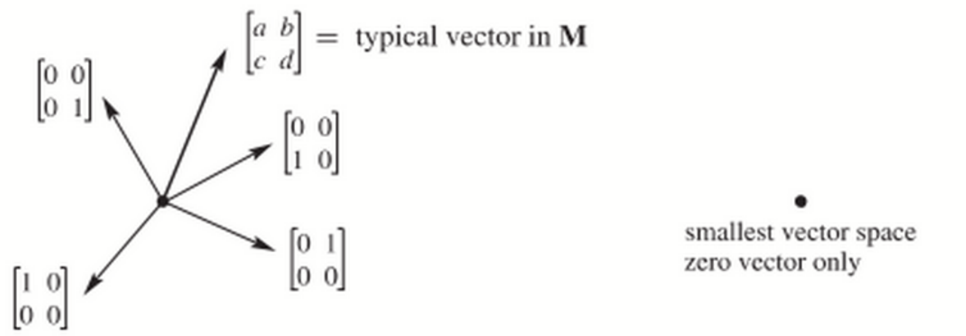


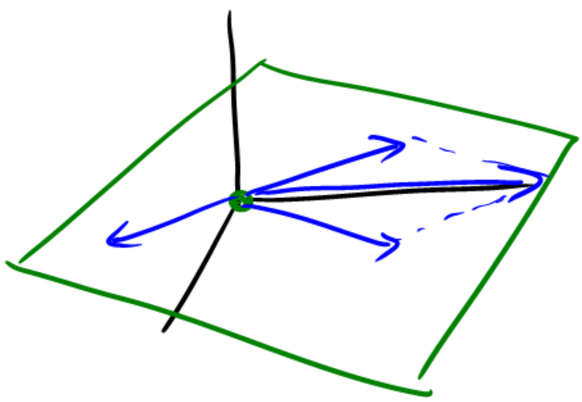
Figure 18: "Four-dimensional" matrix space **M**. The "zero-dimensional" space **Z**.

Subspaces

vector space inside a vector space

Ex: subspaces of R^3

A plane through origin



A line through origin

$\boxed{\text{Def}}$ A subspace nonempty subset of a vector space is a set of vectors (including $\underline{0}$) that satisfies: $\forall\ \underline{u}, \underline{w}$ in the subspace $\forall$ scalar c

(i) $\underline{u} + \underline{w}$ is in the subspace

(ii) $c\underline{u}$ " " " "

(closed under all lin. comb.)

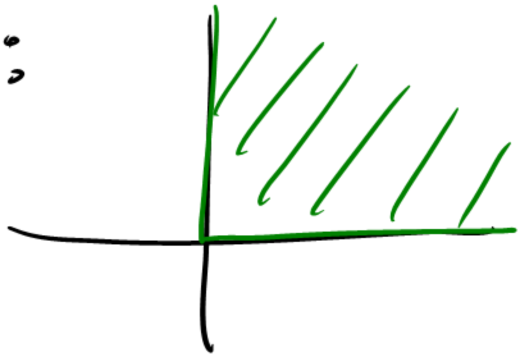
(ADD & MUL follows from the host space $\Rightarrow$ 8 rules are satisfied only need to worry about lin. comb.)

$\boxed{\text{Fact}}$ Every subspace contains $\underline{0}$

(follows from (ii) with $c = 0$)

Not a subspace

Ex 1:

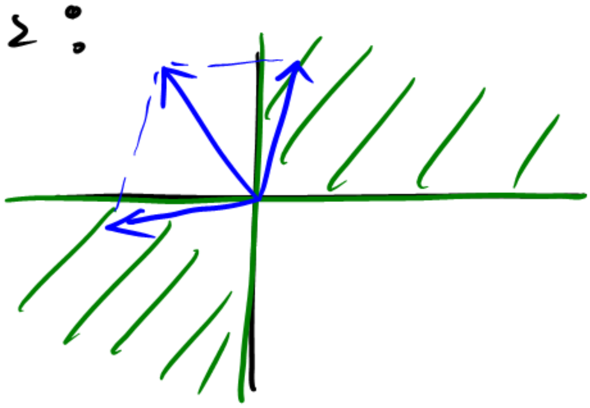


quarter-plane

(i) ok

(ii) x

Ex 2:



(i) x

(ii) ok

$\boxed{\text{Fact}}$ A subspace containing $\underline{u}$ & $\underline{w}$ must contains all lin. comb. $c\underline{u} + d\underline{w}$ (smallest subspace containing $\underline{u}$ & $\underline{w}$ is the set of all comb. of $\underline{u}$ & $\underline{w}$)

Recall: P (Any plane through $\underline{0}$)

L (Any line through $\underline{0}$)

are subspaces in R^3

Q: Is $P \cup L$ a subspace?

No! (i) fails

Q: Is $P \cap L$ a subspace?

Yes!

In general, for any subspaces S & T

$S \cap T$ is also a subspace

(If $\underline{u}, \underline{w}$ in $S \cap T$, $\underline{u}+\underline{w}$ in S

$\underline{u}+\underline{w}$ in T $\Rightarrow$ $\underline{u}+\underline{w}$ in $S \cap T$, similarly for $c\underline{u}$)

Column space of A (Important subspace)

$\boxed{\text{Def}}$ The col. space of A is the vector space made up of all possible lin. comb. of col.s of A (notation: $C(A)$)

Solving $A\underline{x} = \underline{b}$

Q: Given a matrix A, for what vector $\underline{b}$ does $A\underline{x} = \underline{b}$ have a sol.?

Ex:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

(row picture)

(4 eqns. 3 unknowns $\Rightarrow A\underline{x} = \underline{b}$ does not have a sol. for every choice of $\underline{b}$ $\Rightarrow$ only for some $\underline{b}$)

Q: What are those $\underline{b}$?

$\underline{b}$ must be a lin. comb. of col.s of A $\Rightarrow \underline{b} \in C(A)$

(col. picture)

(Another perspective: only 3 col. vectors cannot fill the entire 4D space $\Rightarrow$ some $\underline{b}$ cannot be expressed as lin. comb. of col.s of A)

$\boxed{\text{Fact}}$ The system $A\underline{x} = \underline{b}$ is solvable iff $\underline{b}$ is in the col. space of A

When $\underline{b} \in C(A)$, $\underline{b}$ is a lin. comb. of col.s of A

i.e., $\underline{b} = x_1 \underline{a_1} + \cdots + x_n \underline{a_n}$ for some $x_1 \cdots x_n$

the comb. give you sol. to

$A\underline{x} = \underline{b}$

Back to Ex:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Q: What can we say about $C(A)$?

col. vectors lin dependent or indep.?

or Does each col. contribute sth. new to the subspace?

col. 3 = col. 1 + col. 2 (lin. dependent)

($C(A)$ is 2D subspace of R^4)

In general,

$A_{m \times n}$: n col.s, each with m dim.

$\Rightarrow$ $C(A)$ is a subspace of R^m (not R^n)

Q: Is $C(A)$ really a subspace?

Yes! If $\underline{b}, \underline{b}' \in C(A)$, $\underline{b}, \underline{b}'$ are comb. of col.s of A $\Rightarrow$ $\underline{b} + \underline{b}'$ still comb. of col.s of A $\Rightarrow$ $c\underline{b}$ still comb. of. col.s of A

(or $A\underline{x} = \underline{b}$, $A\underline{x}' = \underline{b}' \Rightarrow A(\underline{x}+\underline{x}') = \underline{b}+\underline{b}'$

$A(c\underline{x}) = c\underline{b}$)

Recall: all comb. of $\underline{u}$, $\underline{w}$ is the smallest subspace containing $\underline{u}$, $\underline{w}$

Notation: for a vector space V

S = set of vectors in V

SS = all comb. of vectors in S

(span of S : smallest subspace containing S)

Note:

The smallest possible col. space $A = \underline{0}$

(only contain $\underline{0}$)

The largest " " " R^m

(Ex: $C(I)$ or any nonsingular $m \times m$ matrix)

(We can use elimination to solve it)

(This is more general than Ch. 1. Now, we allow singular matrices & rectangular matrices of any shape)