EE203001 SBS # 8

Elimination = Factorization : A = LU

$$A \xrightarrow[\text{Steps}]{\text{Elimination}} U$$

or $EA = U \Rightarrow A = E^{-1}U = LU$

$(A \to E_{21}A \to E_{31}E_{21}A \to \cdots \to U)$

A 2x2 Ex

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$\Rightarrow A = E_{21}^{-1}U = \underset{L}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A$$

If no row change, 3x3 Ex

$$(E_{32}E_{31}E_{21})A = U$$

$$\Rightarrow A = (E_{32}E_{31}E_{21})^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U \quad (= LU) \quad \text{—— (1)}$$

Note 1 :

every inverse matrix $E_{21}^{-1}, E_{31}^{-1}, E_{32}^{-1}$ is lower triangular with off-diagonal entry l_{ij} to undo $-l_{ij}$ for E_{ij}

Note 2:

Eqn (1) shows

$$(E_{32}E_{31}E_{21})A = U \Rightarrow A = (\underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L})U$$

Also lower-triangular

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$ (determined exactly by l_{ij})

$\boxed{\text{Fact}}$ If no row change, U has pivots on its diagonal, L has all 1's on its diagonal & l_{ij} below the diagonal

Ex: $E_{31} = I$

$$\underset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}} \underset{E_{21}}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix}}$$

(row 2^{new} = row 2 − 2 · row 1)

row 3 − 5 · row 2^{new} (starting from top)

= row 3 − 5 (row 2 − 2 · row 1)

= row 3 − 5 · row 2 + 10 · row 1

But $L = (E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$

$$= E_{21}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

(row 3^{new} = row 3 + 5 · row 2)

$\downarrow$

(bottom up)

(row2 + 2 · row 1) (does NOT involve row3^{new})

More generally,

$$E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$$

(row3 of U = row3 of A

$- l_{31}$ (row 1 of U)

$- l_{32}$ (row 2 of U)

$\Rightarrow$ row 3 of A = (row 3 + l_{31} row1 + l_{32} · row 2) of U)

row3 + l_{32} · row2

row3 + l_{31} · row 1

$\Rightarrow$ row3^{new} = row 3 + l_{32} · row2 + l_{31} · row1

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ \boxed{l_{31} \quad l_{32} \quad 1} \end{bmatrix}$$

Factor out diagonal matrix

$$U = \underbrace{\begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \\ & & & d_n \end{bmatrix}}_{D} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdots \\ & 1 & u_{23}/d_2 & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

$\Rightarrow A = LDU$

$$\left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right)$$

Q: When do we use LU?

Most computer code use LU to solve $A\underline{x} = \underline{b}$

One square system = Two triangular systems

Step 1 : Factor $A = LU$ (get L for free)

Step 2 : solve $\underline{b}$ using L

(Solve $L\underline{c} = \underline{b}$, then solve $U\underline{x} = \underline{c}$)

(forward & backward substitution)

($L(U\underline{x}) = \underline{b} \Rightarrow A\underline{x} = \underline{b}$)

Ex:
$$\begin{aligned} u + 2v &= 5 \\ 4u + 9v &= 21 \end{aligned} \Rightarrow \begin{aligned} u + 2v &= 5 \\ v &= 1 \end{aligned}$$

or $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$L\underline{c} = \underline{b} \Rightarrow \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\underline{c} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \Rightarrow \underline{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$U\underline{x} = \underline{c} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\underline{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\Rightarrow$ back sub. $\Rightarrow \underline{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Cost of Elimination

For a $n \times n$ matrix, To produce zeros below the first pivot

need $\sim n^2$ mul. & n^2 substraction

(Eg. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$

in fact $n(n-1)$)

Next stage clears out 2^{nd} col. below 2^{nd} pivot $\sim (n-1)^2$ mul & sub.

$\vdots$

To reach U, need $\sim n^2 + (n-1)^2 + \cdots + 1^2$

$$= \frac{1}{3} n (n+\tfrac{1}{2})(n+1) \simeq \frac{1}{3} n^3$$

Q: How about right side b?

Step 1: substract multiples of b_1 from $b_2, \cdots, b_n$ $(n-1)$ mul & sub

Step 2: substract multiples of b_2 from $b_3, \ldots, b_n$ $(n-2)$ mul & sub

$\vdots$

$$(n-1) + (n-2) + \cdots + 1 + 1 + 2 + \cdots + n = n^2$$

(small compared with $\frac{1}{3} n^3$)

Back substitution

compute x_n 1 ''

'' x_{n-1} 2 ''

$\vdots$

n

Q: What if there are row exchanges?

Use permutation matrix P

Transposes & Permutations

Transpose

$(A^T)_{ij} = A_{ji}$ (exchange row & col.)

Ex:

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$

(transpose of lower triangular is upper triangular)

Rules

sum: $(A+B)^T = A^T + B^T$

product: $(AB)^T = B^T A^T$

inverse: $(A^{-1})^T = (A^T)^{-1}$

$(AB)^T = B^T A^T$

pf: Start with $A\underline{x} = [\underline{a_1} \cdots \underline{a_n}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$= x_1 \underline{a_1} + \cdots + x_n \underline{a_n}$ combine col. of A

$\Rightarrow (A\underline{x})^T = x_1 \underline{a_1}^T + \cdots + x_n \underline{a_n}^T$

$$\underline{x}^T A^T = [x_1 \cdots x_n] \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_n^T \end{bmatrix}$$ combine row of A^T

$$= x_1 \underline{a}_1^T + \cdots + x_n \underline{a}_n^T$$

$$\Rightarrow (A\underline{x})^T = \underline{x}^T A^T$$

For $B = [\underline{x}_1 \ \underline{x}_2 \cdots \underline{x}_n]$

$$(AB)^T = [A\underline{x}_1 \ A\underline{x}_2 \cdots A\underline{x}_n]^T = \begin{bmatrix} (A\underline{x}_1)^T \\ \vdots \\ (A\underline{x}_n)^T \end{bmatrix}$$

$$= \begin{bmatrix} \underline{x}_1^T A^T \\ \vdots \\ \underline{x}_n^T A^T \end{bmatrix} = B^T A^T$$

Can extend to 3 or more factors:

$$(ABC)^T = C^T B^T A^T$$

$(A^{-1})^T = (A^T)^{-1}$

pf: $AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T = I$

$$\Rightarrow (A^{-1})^T A^T = I \Rightarrow (A^{-1})^T = (A^T)^{-1}$$ (left inverse)

Similarly for right inverse $(A^{-1}A = I)$

$\Rightarrow A^T$ is invertible iff A is invertible

Symmetric matrix

$A^T = A$ or $a_{ji} = a_{ij}$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A^T$

Note: the inverse of a symmetric matrix is also symmetric

$\left((A^{-1})^T = (A^T)^{-1} = A^{-1} \text{ if } A \text{ symmetric} \right)$

Symmetric product

$R^T R$ is always symmetric for any R

$\left((R^T R)^T = R^T (R^T)^T = R^T R \right)$

(For symmetric A, $A = LDU \Rightarrow A = LDL^T$)

Permutation

Def A permutation matrix P has the rows of the identity I in any order

Ex: 3×3 permutation matrices

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \quad P_{32} P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, \quad P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, \quad P_{21} P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

there are $n!$ permutation matrices of order n

Fact $P^{-1} = P^T$

$$\left(P^T P = \begin{bmatrix} \underline{p_1}^T \\ \vdots \\ \underline{p_n}^T \end{bmatrix} \begin{bmatrix} \underline{p_1} & \cdots & \underline{p_n} \end{bmatrix} = I \Rightarrow P^{-1} = P^T \right.$$

$$\left. \because \underline{p_i}^T \underline{p_j} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases} \right)$$

Q: What if there are row exchanges?

$$PA = LU$$

put all rows of A in right order

If A is invertible, $PA = LU$ s.t.

U has full sets of pivots

Ex:

$$\underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \to \underset{PA}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \to \underset{l_{31} = 2}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}}$$

$$\to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow PA = LU$$

$$l_{32} = 3 \quad \underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} A = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$$