Inverse Matrices

Def The matrix A is invertible (non-singular) if $\exists A^{-1}$ s.t. $A^{-1}A = I$ & $AA^{-1} = I$

Note

Solving $A\underline{x} = \underline{b}$ is the same as finding A^{-1}!

$(A^{-1}(A\underline{x}) = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b})$

(NOT all matrices have inverses)

Note 1:

The inverse exists iff elimination produces n pivots (row exchanges allowed)

($A\underline{x} = \underline{b}$ is solvable)

Note 2:

left inverse = right inverse

$BA = I$ $\qquad$ $AC = I$

$[B(AC) = (BA)C \Rightarrow BI = IC \Rightarrow B = C]$

Note 3:

If A is invertible, $A\underline{x} = \underline{b}$ only has

One sol. : $\underline{x} = A^{-1}\underline{b}$

$$\left(A^{-1}A\underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b} \right)$$

Note 4 :

Suppose $\exists$ a nonzero vector $\underline{x}$ s.t. $A\underline{x} = \underline{0}$

$\Rightarrow A^{-1}$ does NOT exist

(not possible to have $A^{-1}(\overset{\underline{0}}{\overbrace{A\underline{x}}}) = \underline{x}$)

(If A invertible , $A\underline{x} = \underline{0}$ can only have zero sol. , i.e., $\underline{x} = \underline{0}$)

Note 5 :

A 2×2 matrix is invertible iff $\underset{\text{determinant}}{ad-bc} \neq 0$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note 6 :

A diagonal matrix has an inverse iff no diagonal entries are zero

$$A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$$

Ex :

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

fail Note 1 (only one pivot)

Note 4 ($A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \underline{0}$)

Note 5 ($ad-bc = 0$)

Inverse of a product

$\boxed{\text{Fact}}$

If A, B both invertible $\Rightarrow$ AB invertible

& $(AB)^{-1} = B^{-1}A^{-1}$

(pf : $(B^{-1}A^{-1})(AB) = B^{-1}IB = I$)

(can be applied to 3 or more products

$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ ...)

Ex : Inverse of elimination matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(row 2 − 5 · row 1) (row 2 + 5 · row 1)

(chk $EE^{-1} = I$)

For square matrices, (left inverse is automatically a righ inverse)

if $AB = I \Rightarrow BA = I$ ($B = A^{-1}$)

Gauss - Jordan Elimination

For 3×3 matrix

$$AA^{-1} = A[\underline{x}_1\ \underline{x}_2\ \underline{x}_3] = [\underline{e}_1\ \underline{e}_2\ \underline{e}_3] = I$$

$\Rightarrow$ solve 3 systems of eqns

$$A\underline{x_1} = \underline{e_1}, \quad A\underline{x_2} = \underline{e_2}, \quad A\underline{x_3} = \underline{e_3}$$

Augmented matrix:

$$[\,A \mid I\,] \longrightarrow [\,I \mid E\,] \quad (A \to U \to I)$$

$$\Big(\Rightarrow E\,[\,A \mid I\,] = [\,I \mid E\,]$$

$$\Rightarrow [\,EA \mid E\,] = [\,I \mid E\,] \Rightarrow E = A^{-1}\Big)$$

Ex:

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$[\,K \mid I\,] \longrightarrow [\,I \mid K^{-1}\,]$$

(p. 84)