EE 203001 SES # 6

Rules for matrix operations

Addition

$A_{(m \times n)}$: m rows , n columns

$B_{(p \times q)}$: p rows , q columns

Q: Can you do A + B ?

Only when $m = p$, $n = q$

$\Rightarrow$ two matrices are of same size

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \end{bmatrix} \quad (\checkmark)$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad (\times)$$

Multiplication

Q: Can you do AB ?

If A has n col.s , we can do AB only when B has n rows

$$\Rightarrow A_{(m \times n)} B_{(n \times p)} = C_{(m \times p)}$$

[chk of dim is important to trace errors]

Four diff. ways of thinking $AB = C$

Standard (rows x col.s) inner product

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$

$$\begin{bmatrix} & * & \\ a_{i1} & a_{i2} \cdots & a_{i5} \\ & * & \\ & * & \end{bmatrix} \begin{bmatrix} * & * & b_{1j} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

A is 4 by 5 $\qquad$ B is 5 by 6 $\qquad$ AB is 4 by 6 $= C$

or $C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Ex 1

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$$

(dim. chk: square matrices can be multiplied iff they are of same size)

(if $n \times n$: involves n^2 dot products

each dot product $= n$ multiplications

$\Rightarrow$ total n^3 multiplications)

Ex 2

$$\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 + 6 = 8$$ (inner product)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix}$$ (outer product)

Columns

$$C = AB = A[b_1 \, b_2 \cdots b_p] = [Ab_1 \; Ab_2 \cdots Ab_p]$$

each col. of C is Ab_i (lin. comb. of col. of A)

$\Rightarrow$ each col. of C is a lin. comb. of col.s of A

Rows

$$\begin{bmatrix} \underline{c}_1^T \\ \underline{c}_2^T \\ \vdots \\ \underline{c}_m^T \end{bmatrix} = \begin{bmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \vdots \\ \underline{a}_m^T \end{bmatrix} B = \begin{bmatrix} \underline{a}_1^T B \\ \underline{a}_2^T B \\ \vdots \\ \underline{a}_m^T B \end{bmatrix}$$

each row of C is $\underline{a}_i^T B$ (lin. comb. of rows of B)

$\Rightarrow$ each row of C is a lin. comb. of rows of B

Column times row

each is a $m \times p$ matrix

$$[\underline{a}_1 \; \underline{a}_2 \cdots \underline{a}_n] \begin{bmatrix} \underline{b}_1^T \\ \underline{b}_2^T \\ \vdots \\ \underline{b}_n^T \end{bmatrix} = \underline{a}_1 \underline{b}_1^T + \underline{a}_2 \underline{b}_2^T + \cdots + \underline{a}_n \underline{b}_n^T$$

Ex

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\parallel$$

$$\begin{bmatrix} \boxed{3 \quad 2} \\ \boxed{6 \quad 4} \end{bmatrix}$$

$(3, 2)$ lies in the same line as $(6, 4)$

$\Rightarrow$ row space is a line

similarly, col. space is also a line

Blocks

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

here $C_1 = A_1 B_1 + A_2 B_3$

Ex Elimination by blocks

$$A = \begin{bmatrix} 1 & \times & \times \\ 3 & \times & \times \\ 4 & \times & \times \end{bmatrix}$$

one at a time

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$E = E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$EA = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -3 & 1 & 0 \\ -4 & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} 1 & x & x \\ \hline 3 & x & x \\ 4 & x & x \end{array}\right] = \left[\begin{array}{c|cc} 1 & x & x \\ \hline 0 & x & x \\ 0 & x & x \end{array}\right]$$

$$\left(\begin{bmatrix} -3 \\ -4 \end{bmatrix} 1 + I \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

(Schur complement)

$$\left(\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right] \right)$$

(chk: $-CA^{-1}A + C = 0$, $-CA^{-1}B + D$ ")

The Laws for matrix operations

For addition

$$A + B = B + A \qquad \text{commutative}$$

$$c(A+B) = cA + cB \qquad \text{distributive}$$

$$A + (B+C) = (A+B) + C \qquad \text{associative}$$

For multiplication

$$AB \neq BA \qquad \text{commutative broken!}$$

$$C(A+B) = CA + CB \qquad \text{distributive from left}$$

$$(A+B)C = AC + BC \qquad \text{" " right}$$

$$A(BC) = (AB)C \qquad \text{associative}$$

$AB \neq BA$

Obvious if A, B not square

$A_{(m\times n)} B_{(n\times p)} = AB_{(m\times p)}$

$B_{(n\times p)} A_{(m\times n)}$ (not legal if $p \neq m$)

$B_{(n\times m)} A_{(m\times n)} = BA_{(n\times n)}$ ($p = m$)

($AB_{(m\times m)} \neq BA$ if $m \neq n$)

Even if both square,

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Exception

$AI = IA$ (only cI commute with other matrices)

$A(B+C) = AB + AC$

$A(\underline{b}+\underline{c}) = A\underline{b} + A\underline{c}$ (proved a col. at a time)

Powers

$A^p = AA \cdots A$ (p factors)

$(A^p)(A^q) = A^{p+q}$. $(A^p)^q = A^{pq}$