The geometry of linear eqns

Central problem of linear algebra

Solving a system of linear eqns!

Ex

$$x - 2y = 1$$
$$3x + 2y = 11 \quad (2 \text{ eqns}, 2 \text{ unknowns})$$

[We can have 3 diff. views on this!]

Row Picture

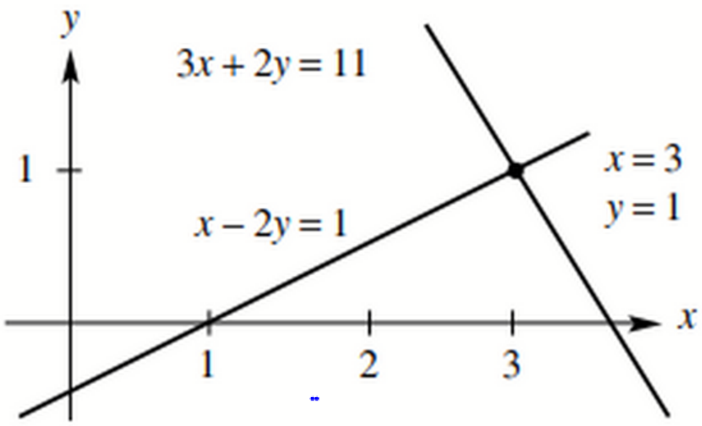

Figure 11: *Row picture*: The point (3, 1) where the lines meet is the solution.

Solution $x = 3,\ y = 1$ is where the two lines meet

($x = 3,\ y = 1$ is the point that satisfies both linear eqns)

(plug in & see)

Column Picture

See the same lin. equs as vector equs

$$\Rightarrow x\begin{bmatrix}1\\3\end{bmatrix} + y\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix} = \underline{b}$$

$$\begin{array}{cc} \underline{c} & \qquad \underline{d} \end{array}$$

$$\Rightarrow x\underline{c} + y\underline{d} = \underline{b}$$

(lin. comb. of two col. vector gives $\underline{b}$)

Now we need to find scalars x & y s.t. x copies of $\begin{bmatrix}1\\3\end{bmatrix}$ + y copies of $\begin{bmatrix}-2\\2\end{bmatrix}$ equals the vector $\begin{bmatrix}1\\11\end{bmatrix}$

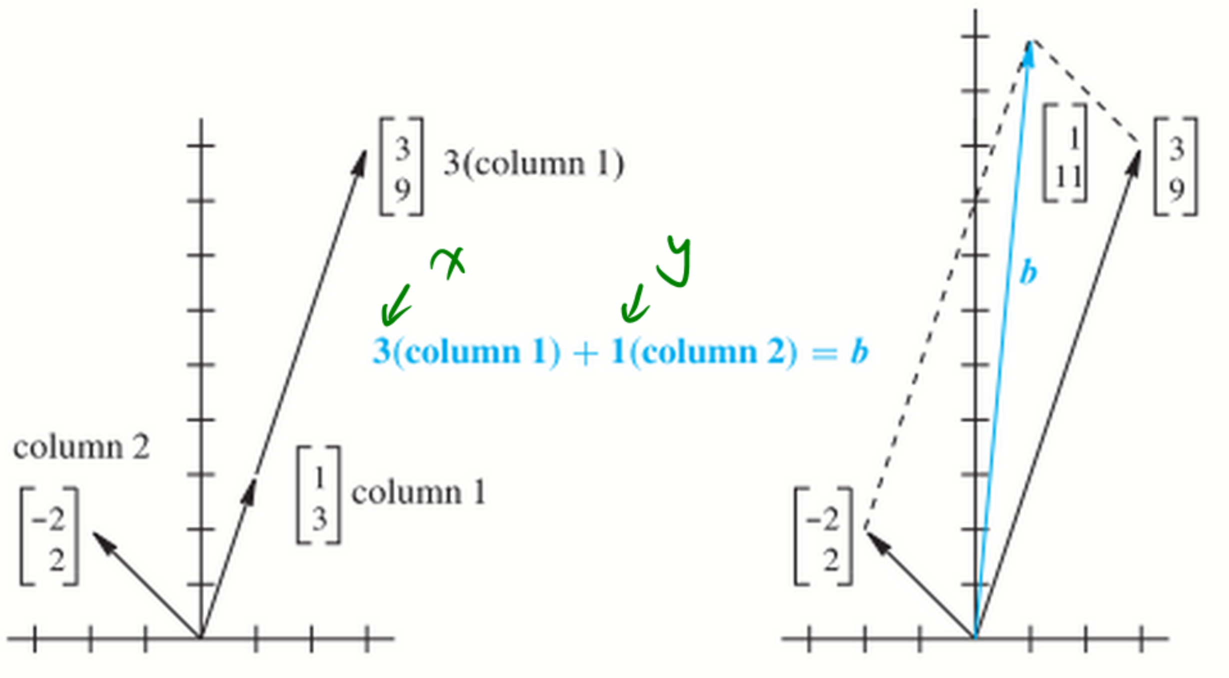

Figure 12: *Column picture*: A combination of columns produces the right side (1,11).

Linear comb.

$$3\begin{bmatrix}1\\3\end{bmatrix}+1\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

(same sol. $x=3$, $y=1$ but diff. views)

Matrix Picture

$$\begin{aligned}x-2y&=1\\3x+2y&=11\end{aligned}\Rightarrow \underbrace{\overset{A}{\begin{bmatrix}1&-2\\3&2\end{bmatrix}}}_{\text{coeff. matrix}}\underbrace{\overset{\underline{x}}{\begin{bmatrix}x\\y\end{bmatrix}}}_{\text{vector of unknowns}}=\overset{\underline{b}}{\begin{bmatrix}1\\11\end{bmatrix}}$$

$\Rightarrow A\underline{x}=\underline{b}$

Matrix multiplication

Method 1

$$\begin{bmatrix}1&-2\\3&2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=3\begin{bmatrix}1\\3\end{bmatrix}+1\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

(based on col. picture)

Method 2

$$\begin{bmatrix}1&-2\\3&2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}1\cdot3-2\cdot1\\\underbrace{3\cdot3+2\cdot1}_{\text{dot product}}\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix}$$

(based on row picture)

Three eqns in three unknowns

$$Ax = b \iff \begin{array}{rcl} x + 2y + 3z &=& 6 \\ 2x + 5y + 2z &=& 4 \\ 6x - 3y + z &=& 2 \end{array}$$

Row Picture

(3 eqns + 3 unknowns : usually one sol.)

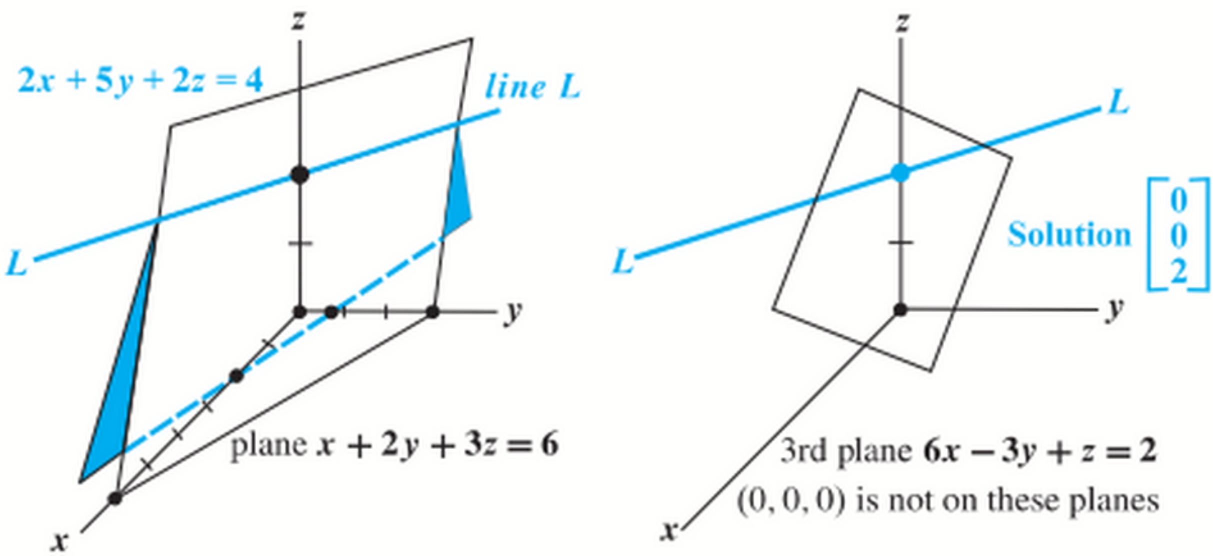

Figure 13: *Row picture*: Two planes meet at a line, three planes at a point.

(Note: $x + 2y + 3z = 0$ passes through origin

$x + 2y + 3z = 6$ does NOT

but is parallelly shifted)

(Sol. is difficult to visualize & find)

Column Picture

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

(Very easy to see that $x=0, y=0, z=2$ is the sol.)

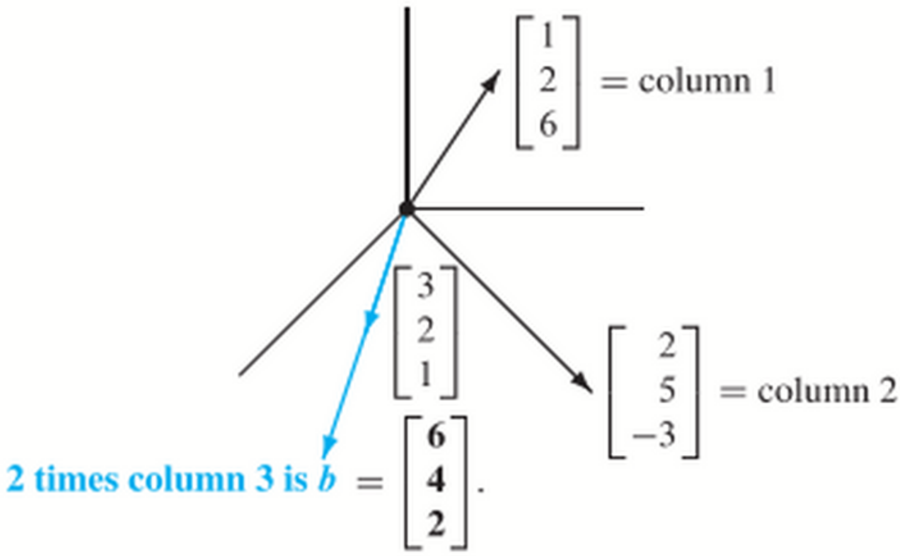

Figure 14: *Column picture*: $(x, y, z) = (0, 0, 2)$ because $2(3, 2, 1) = (6, 4, 2) = \boldsymbol{b}$.

$$\left(0\begin{bmatrix}1\\2\\6\end{bmatrix} + 0\begin{bmatrix}2\\5\\-3\end{bmatrix} + 2\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix} = \underline{b} \right)$$

Matrix Picture

$$\underbrace{\begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix}6\\4\\2\end{bmatrix}}_{\underline{b}}$$

Multiplication by rows

$$Ax = \begin{bmatrix}(\text{row } 1)\cdot \underline{x}\\(\text{row } 2)\cdot \underline{x}\\(\text{row } 3)\cdot \underline{x}\end{bmatrix}$$

Multiplication by columns

$$A\underline{x} = x\,(\text{col. }1) + y\,(\text{col. }2) + z\,(\text{col. }3)$$

Identity matrix

ones on the "main diagonal"

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I\underline{x} = \underline{x}$$

Matrix notation

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{bmatrix}$$

Linear independence

Q: Given a matrix A, can we solve

$$A\underline{x} = \underline{b}$$

for every possible vector $\underline{b}$?

From col. picture

Q: Do lin. comb. of col.s of A fill the entire space? (2D or 3D)

If not, we say A is singular

$\Rightarrow$ col.s of A are linearly dependent

(For 2D, lin. comb. of col. vectors lies on a point, or a line)
(0,0)

(For 3D, lin comb. of col. vectors lies on a point, line, or plane)

worked ex. 2.1 A 2.1 B