EE 203001 SES # 14

New vector space

All 3x3 matrices form a vector space M

(We can add matrices , mutiply by scalars & there is a zero matrix

($A+B$, cA) (not AB for now)

(All 8 rules are satisfied)

Subspaces

- All upper triangular matrices (U)
- All symmetric matrices (S)
- All diagonal matrices (D)

Note: $D = U \cap S$

Q: What is the dim. of D ?

$\dim D = 3$

basis: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q: What is the dim of M?

$\dim M = 9$

basis: (standard)

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix},\ \cdots,\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

Very similar to R^9 just arrange in a matrix form

Q: What is the dim. of the subspace S?

$\text{Dim } S = 6$ (pick 3 diagonal elements + 3 in the upper right)

(lower left determined by upper right)

Basis: (Also basis for M)

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&&\\&1&\\&&0\end{bmatrix},\begin{bmatrix}0&&\\&0&\\&&1\end{bmatrix}}_{\text{Also basis for M}},\begin{bmatrix}0&1&\\1&0&\\&&0\end{bmatrix},$$

$$\begin{bmatrix}0&&\\&0&1\\&1&0\end{bmatrix},\begin{bmatrix}&0&1\\0&0&0\\1&0&0\end{bmatrix}$$

(NOT basis for M)

Q: What is the dim. of the subspace U?

$\dim U = 6$ (3 diagonal + 3 upper right)

Basis: (diff. from S)

$$\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$ (happens to be a subset of basis of M)

Other subspaces

$S \cap U$ = symmetric & upper triangular
$= D$

$\dim(S \cap U) = 3$

$S \cup U$ = symmetric or upper triangular

(NOT a subspace since a symmetric matrix + a upper triangular matrix is NOT in $S \cup U$ in general)

(Analogy: two lines in $\mathbb{R}^2$ is NOT a subspace. Need to fill in between them)

Instead,

$S+U$ = any element of S + any element of U

(Sum subspace)

= All $3\times3 = M$

$$\left(\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ & 0 & 2 \\ & & 0 \end{bmatrix} \right)$$

$\dim(S+U) = 9$

In general,

$$\underbrace{\dim S}_{6} + \underbrace{\dim U}_{6} = \overbrace{\dim(S+U)}^{9} + \underbrace{\dim(S\cap U)}_{3}$$

Differential eqns as a vector space

$\dfrac{d^2y}{dx^2} + y = 0$, sol. to this eqn is an element of the nullspace

possible sol.s:

$y = \underbrace{\cos x, \sin x}_{\text{basis}}, e^{ix}$ (special sol.s)

Complete sol: $y = c_1\cos x + c_2 \sin x$

$\dim\left(\begin{matrix}\text{sol.}\\ \text{space}\end{matrix}\right) = 2$ (since this is a 2^{nd}-order eqn)

(Don't look like vectors, but we can build a vector space from it since we can add & multiply by a scalar)

Rank one matrices

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$$

Q: What is the rank of A?

$\text{rank } A = 1$ (row 2 = 2 · row 1)

$$\dim C(A) = \underset{\substack{\| \\ 1}}{\text{rank}} = \dim C(A^T)$$

or

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \underset{2\times 1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}$$

(each col. is a multiple of col. 1 or each row is a multiple of row 1)

In general, for every rank-1 matrix

$$A = \underline{u}\,\underline{v}^T = \begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} & & \end{bmatrix}$$

(building blocks for more complicated matrix, e.g., a 5×17 rank-4 matrix can be written as comb.

of 4 rank-1 matrices)

(To be dissussed later)

Q: Is subset of rank-1 matrices a subspace?

No, since sum of two rank-1 matrices may NOT be rank-1

Another example

In R^4, the set of all vectors $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

for which $v_1 + v_2 + v_3 + v_4 = 0$

Q: Is this a subspace?

Yes! it contains $\underline{0}$ & closed under

ADD & scalar MUL

($\underline{w} + \underline{v} = \begin{bmatrix} w_1+v_1 \\ w_2+v_2 \\ w_3+v_3 \\ w_4+v_4 \end{bmatrix}$) sum of all components $= 0$)

Q: What is the dim.?

This is the Nullspace of

$A = [1 \ 1 \ 1 \ 1]$

$\text{rank}(A) = 1$

$\Rightarrow \dim N(A) = n - r = 4 - 1 = 3$

Basis :

Find special sol s

$$\left.\begin{array}{l} \text{col.}2 = 1 \cdot \text{col }1 \\ \text{col.}3 = \quad " \\ \text{col.}4 = \quad " \end{array}\right) \Rightarrow \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>Q: What is $C(A)$?</u>

1 pivot & $m = 1 \Rightarrow C(A) = \mathbb{R}^1$

<u>Q: What is $N(A^T)$:</u>

$y^T [1 \ 1 \ 1 \ 1] = [0 \ 0 \ 0 \ 0]$

$\Rightarrow y = 0 \Rightarrow N(A^T) = \{0\}$

(basis is empty set)

<u>Q: What is $C(A^T)$?</u>

$\dim C(A^T) = r = 1$

basis : $[1 \ 1 \ 1 \ 1]$

<u>Chk dim</u>

$\dim C(A^T) + \dim N(A) = 1 + 3 = 4 = n$

$\dim C(A) + \dim N(A^T) = 1 + 0 = 1 = m$