EE203001 SBS #13

The four fundamental subspaces

Any $m \times n$ matrix A determines four subspaces (possibly containing only the zero vector)

Column space, $C(A)$: (in R^m)

All comb. of the col.s of A

Null space, $N(A)$: (in R^n)

All sol.s of $\underline{x}$ of $A\underline{x} = \underline{0}$

Row space, $C(A^T)$: (in R^n)

All comb. of row vectors of A

(same as col. space of $A^T \Rightarrow C(A^T)$)

Left null space, $N(A^T)$: (in R^m)

Null space of $A^T \Rightarrow$ All sol.s of $\underline{y}$ of $A^T \underline{y} = \underline{0}$

($A^T\underline{y} = \underline{0} \Leftrightarrow \underline{y}^T A = \underline{0}^T$ so called left Null space)

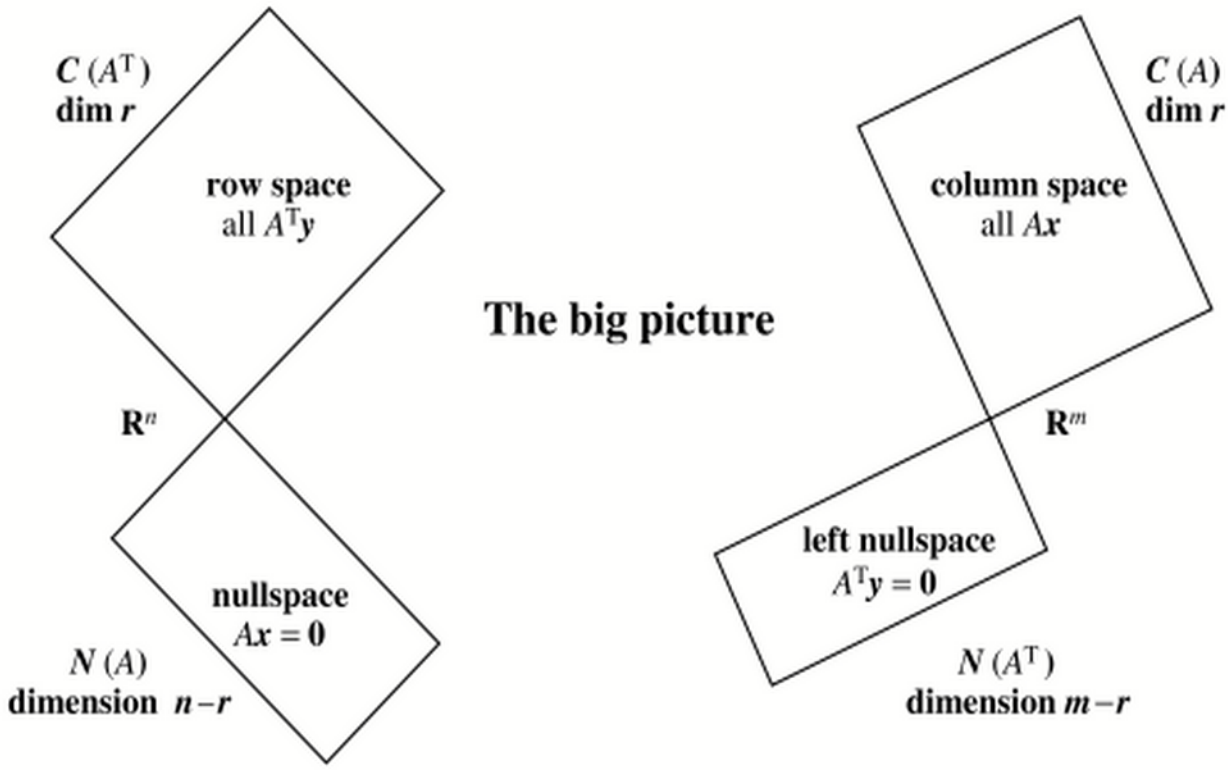

Figure 22: The dimensions of the Four Fundamental Subspaces (for R and for A).

Basis & dimension

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\left(\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \right)$$

Column space $C(A)$

Dim:

$$\dim C(A) = \operatorname{rank}(A)$$
$$= \# \text{ of pivot col.s.}$$

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{rank}(A) = \operatorname{rank}(R) = 2$$

Basis:

the r pivot col.s form a basis for $C(A)$

$$A = \begin{bmatrix} \boxed{\begin{matrix}1\\1\\1\end{matrix}} & \boxed{\begin{matrix}2\\1\\2\end{matrix}} & \begin{matrix}3\\2\\3\end{matrix} & \begin{matrix}1\\1\\1\end{matrix} \end{bmatrix} \to \cdots \to \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \begin{matrix}1\\1\\0\end{matrix} & \begin{matrix}1\\0\\0\end{matrix} \end{bmatrix} = R$$

($C(A) \neq C(R)$, but positions of pivot col.s are the same

$\because R = EA$ or $A = E^{-1}R$

& pivot col.s of R are col.s of I)

(Another example: SES-10, p.8)

<u>Null space</u> $N(A)$

Dim:

$$\dim N(A) = \# \text{ of free col.s of } A$$
$$= \# \text{ of free col.s of } R$$
$$= n - r$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

free col.s   free. col.s

$\Rightarrow \dim N(A) = 4 - 2 = 2$

Basis:

special sol.s to $A\underline{x} = \underline{0}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\left(\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \right)$$

col.3 = 1 · col.1 + 1 · col 2

col.4 = 1 · col 1

$$\Rightarrow \underline{s}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \underline{s}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ (basis)}$$

Row space $C(A^T)$

Dim:

$$\dim C(A^T) = \#\text{ of pivot rows}$$
$$= \#\text{ of pivot col.s} = r$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{bmatrix} = R$$

(# of indep. col.s = # of indep. rows)

Basis:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \end{bmatrix} = R$$

$\Rightarrow R = EA$

so rows of R are comb. of rows of A

reversible $\Rightarrow A = E^{-1}R$

this implies rows of A are comb. of rows of R (only pivot rows)

$\Rightarrow C(A^T) = C(R^T)$

& first r rows of R form the basis of $C(A^T)$

Left nullspace $N(A^T)$

Dim:

matrix A^T has m col.s

From $\dim C(A^T) = r \Rightarrow \text{rank}(A^T) = r$

$\Rightarrow$ # of pivot col.s of $A^T = r$

$\Rightarrow$ # of free col.s of $A^T = m - r$

$\Rightarrow \dim N(A^T) = m - r$

Basis:

Recall: Gauss-Jordan

$$[A_{n\times n} \; I_{n\times n}] \to [I_{n\times n} \; \underbrace{A^{-1}_{n\times n}}_{=E_{n\times n}}]$$

Similarly,

$$[A_{m\times n} \; I_{m\times m}] \to [R_{m\times n} \; E_{m\times m}]$$

$$EA = R$$

(This is how we obtain E directly)

$$\underbrace{\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ \boxed{-1 \quad 0 \quad 1} \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 0} \end{bmatrix}}_{R}$$

Recall:

$$A^T\underline{y} = \underline{0} \iff \underline{y}^T A = \underline{0}^T$$

(so we have $y^T = [-1 \;\; 0 \;\; 1]$)

In general, ($\because$ $m - r = 3 - 2 = 1$, we only need one basis vector)

the bottom $m-r$ rows of E describes lin. dependencies of rows of A since the bottom $m-r$ rows of R are zero

$\Rightarrow$ The bottom $m-r$ rows of E satisfies $\underline{y}^T A = \underline{0}$

$\Rightarrow$ they are basis for $N(A^T)$

<u>Summary</u>

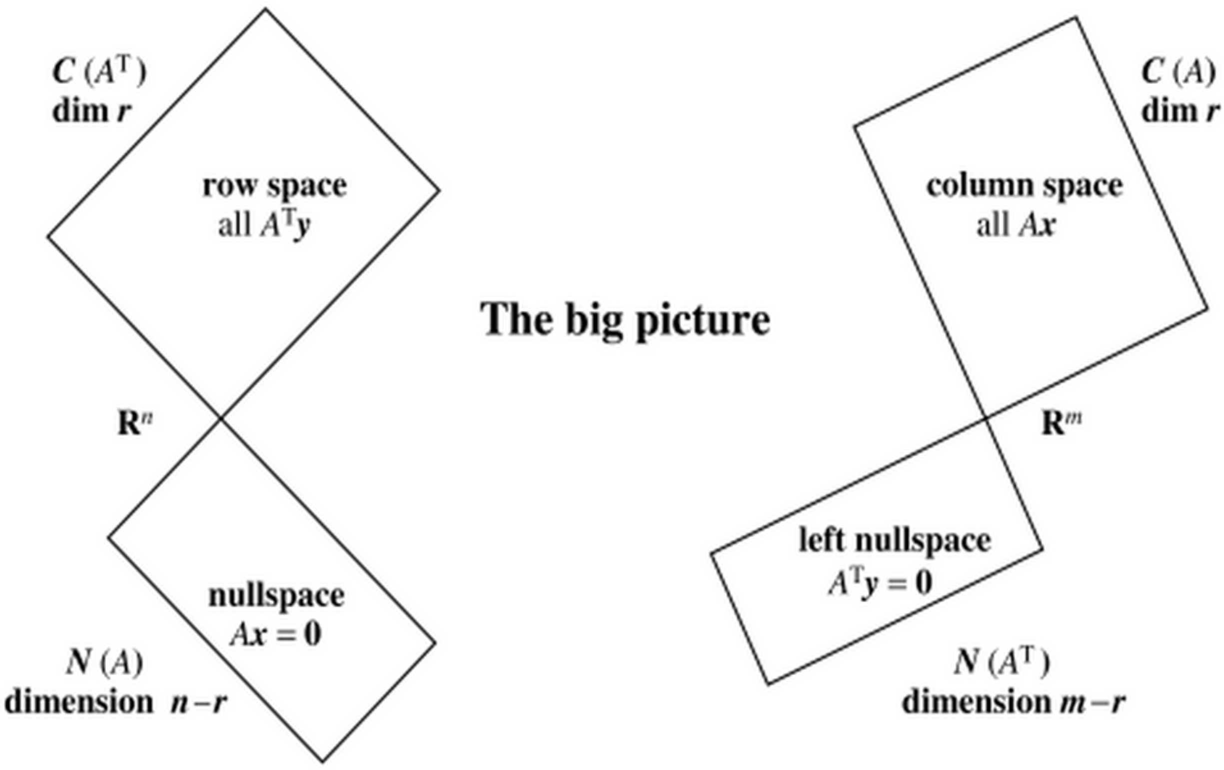

Figure 22: The dimensions of the Four Fundamental Subspaces (for R and for A).

Basis:

$C(A)$ — r pivot col.s of A ($\neq C(R)$)

$N(A)$ — $n-r$ special sol.s are a basis of $N(A)$ & $N(R)$ (same space)

$C(A^T)$ — r pivot rows of R are a basis of $C(A^T)$ & $C(R^T)$ (same space)

$N(A^T)$ — last $m-r$ rows of E are a basis of $N(A^T)$

<u>Fundamental Thm of Linear Algebra</u> (part I)

$C(A)$ & $C(A^T)$ both have $\dim = r$

$\dim N(A) = n-r$, $\dim N(A^T) = m-r$