EE 203001 SES # 12

Independence, basis, and dimension

Recall: (1)

Suppose $A_{m\times n}$ with $m<n$

Then there are nonzero sol. for $A\underline{x}=\underline{0}$

(more unknowns than equ.s)

Reason: A has at least one free var.

$$R=[I\ F] \quad \text{or} \quad R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \quad (n-m)$$

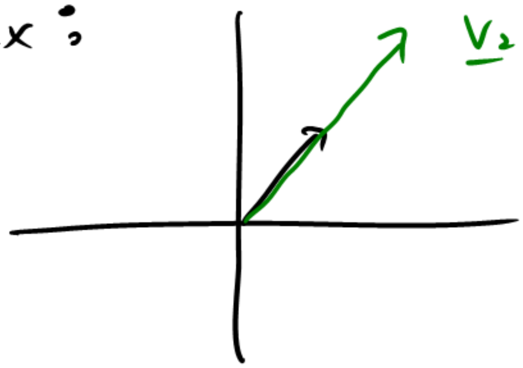
(We will come back to this later)

$\boxed{\text{Def}}$ The vectors $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n$ are lin. indep. if no combination (except the zero comb.) gives zero vector

$$x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_n\underline{v}_n \neq \underline{0}$$

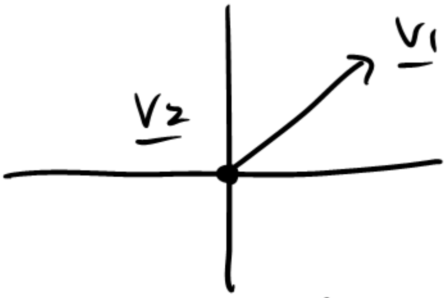
(except $x_1 = x_2 = \cdots = x_n = 0$)

Ex:

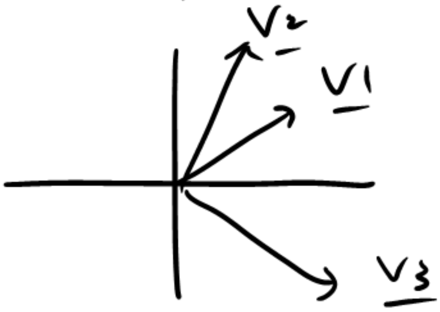


$\underline{v}_2 = 2\underline{v}_1$ (dependent)

$2\underline{v}_1 - \underline{v}_2 = \underline{0}$



$0\,\underline{v}_1 + \underline{v}_2 = \underline{0}$ (dependent)



$\underline{v}_1, \underline{v}_2$ are indep.

Q: How about $\underline{v}_1, \underline{v}_2, \underline{v}_3$?

Back to (1)

$$A = \begin{array}{ccc} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ \end{array} \left[\begin{array}{ccc} 2 & 1 & 3 \\ 1 & 2 & -1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{0}$$

$\Rightarrow$ Whether $A\underline{x} = 0$ has nonzero sol.

is the same as whether $\underline{v}_1, \underline{v}_2, \underline{v}_3$

are lin. indep.

Repeat: When $\underline{v}_1, \cdots, \underline{v}_n$ are col.s of A

They are indep. if $N(A) = \{\underline{0}\}$

(rank $= n$, no free var.s)

" " dependent if $A\underline{x} = \underline{0}$ for some nonzero $\underline{x}$

(rank $< n$, Yes! free var.s)

If $m < n \Rightarrow$ At least $n-m$ free var.s

$\Rightarrow$ col.s of A are lin. dependent

$\Rightarrow \underline{v_1}, \underline{v_2}, \underline{v_3}$ has to be dependent!

(7 dim space, 10 vectors $\Rightarrow m=7, n=10$

$\Rightarrow$ lin. dependent $\because m < n$)

$\boxed{\text{Fact}}$ Any set of n vectors in R^m must be lin. dependent if $m < n$

Spanning a space

$\boxed{\text{Def}}$ Vectors $\underline{v_1}, \cdots, \underline{v_\ell}$ span a space if the space consists of all comb. of these vectors

(Ex: col.s of A spans $C(A)$)

$\boxed{\text{Fact}}$ If $\underline{v_1}, \cdots, \underline{v_\ell}$ span a space S then S is the smallest space that contains these vectors

Col. space

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C(A) = R^2$

Ex: $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \end{bmatrix}, \; C(A) = R^2$

(col.s may be dependent)

Def The row space of a matrix is the subspace of $\mathbb{R}^n$ spanned by the rows

$\Rightarrow$ row space of $A = C(A^T)$

$\Rightarrow$ it's the col. space of A^T

Ex:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \Rightarrow C(A) = \text{plane in } \mathbb{R}^3$$

spanned by 2 vectors

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix} \Rightarrow C(A^T) = \mathbb{R}^2$$

same dim but diff. spaces

(Rows in $\mathbb{R}^n$ spanning the row space

Col.s in $\mathbb{R}^m$ " " col. space)

Basis & dim.

Def A basis for a space is a sequence of vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_d$ with two properties:

1. They are independent
2. They span the space

(Tell us everything we need to know about the space)

Ex: space is R^3

one basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

(standard basis)

Test independence:

Method I: $x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow x_1 = x_2 = x_3 = 0 \Rightarrow$ indep.

Method 2:

$A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ no free var.

$N(A) = \{\underline{0}\}$ or $\left(A\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0} \right)$

$\Rightarrow$ indep.

Q: Is $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$ a basis?

$A = \begin{bmatrix}1&2&3\\1&2&3\\2&5&8\end{bmatrix}$ Do elimination

$\Rightarrow$ first two rows are the same

$\Rightarrow$ only two pivot, one free var.

$\Rightarrow$ dependent

$\Rightarrow$ NOT a basis

In general, n vectors in R^n form a basis if they are col.s of an invertible matrix.

Q: Is $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}$ a basis?

Yes! For a plane S spanned by these vectors in $\mathbb{R}^3$

Q: How many basis do we have for $\mathbb{R}^3$?

Infinitely many!

$\boxed{\text{Fact}}$ Every basis for the space has the same number of basis

(This number is the dim. of the space)

More on basis

$\boxed{\text{Fact}}$ There is only one & only one way to write $\underline{v}$ as a comb. of basis

Reason:

$$\begin{array}{rl} \text{Let} & \underline{v} = a_1\underline{v_1} + \cdots + a_n\underline{v_n} \\ - & \underline{v} = b_1\underline{v_1} + \cdots + b_n\underline{v_n} \\ \hline & \underline{0} = (a_1 - b_1)\underline{v_1} + \cdots + (a_n - b_n)\underline{v_n} \end{array}$$

Since $\underline{v_i}$'s are lin. indep.

$\Rightarrow a_1 - b_1 = 0, \ \ldots, \ a_n - b_n = 0$

$\Rightarrow a_1 = b_1, \ \ldots, \ a_n = b_n$

$\boxed{\text{Fact}}$ The pivot col.s of A are a basis for $C(A)$. The pivot rows of A are a basis for $C(A^T)$. So are the pivot rows of R (not true for col.s)

Ex: $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \to R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ basis for col. space: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ not $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

basis for row space: both $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

col. 3 = col. 1 + col. 2 . col 4 = col 1

col. 1 & 2 are indep.

$\Rightarrow$ basis for $C(A)$ are $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$\boxed{\text{Fact}}$ For any matrix A

$$\begin{aligned} \text{rank}(A) &= \#\text{ of pivot col.s of } A \\ &= \text{dim. of } C(A) \end{aligned}$$

(Matrix has a rank, not a dim. subspace has a dim., not a rank)

Another basis for $C(A)$:

col.1 & col.3 , col.2 & col.3 , ...

(infinitely many basis but dim = 2)

Q: How about $N(A)$?

Special sol.s $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$\Rightarrow$ dim. $= 2$

$\boxed{\text{Fact}}$ For any matrix A

dim. of $N(A)$ = # of free var.s

$= n - r$

(dim. of $N(A) = 4 - 2 = 2$)