EE 203001 SES # 11

Solving $A\underline{x} = \underline{b}$: row reduced form R

Again,

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

(row 3 = row 1 + row 2 $\Rightarrow b_3 = b_1 + b_2$

otherwise, no sol. for $A\underline{x} = \underline{b}$)

Q: How to find sol.?

Also use Elimination!

$$A\underline{x} = \underline{b} \longrightarrow U\underline{x} = \underline{c} \rightarrow R\underline{x} = \underline{d}$$

Elimination with augmented matrix

$$[A\ \underline{b}] = \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix}$$

$$\longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix} \rightarrow [U\ \underline{c}]$$

(need $0 = 0$ for last row

$\Rightarrow b_3 - b_2 - b_1 = 0$)

Ex: $\underline{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

$\Updownarrow$ Equivalent

Recall: $A\underline{x} = \underline{b}$ is solvable iff $\underline{b} \in C(A)$

Complete solution

Step 1 : chk eqn is solvable

Step 2 : find a particular solution $\underline{x}_p$

Step 3 : complete sol = particular sol. + all vectors in $N(A)$ $\underline{x}_n$

A Particular sol.

$$[A\ \underline{b}] \rightarrow [U\ \underline{c}] \rightarrow [R\ \underline{d}]$$

set all free var. = 0

$$[U\ \underline{c}] = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free col.s $\Rightarrow$ set $x_2 = x_4 = 0$

$\Rightarrow \quad x_1 + 2x_3 = 1$

$2x_3 = 3 \Rightarrow x_3 = 3/2 \Rightarrow x_1 = -2$

$$\Rightarrow \underline{x}_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

Using $[R\ \underline{d}]$

$$[U\ \underline{c}] = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[R\ \underline{d}] = \begin{bmatrix} 1 & 2 & 0 & -2 & -2 \\ 0 & 0 & 1 & 2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R\underline{x}_p = \begin{bmatrix} \boxed{\begin{matrix}1\\0\end{matrix}} & 2 & \boxed{\begin{matrix}0\\1\end{matrix}} & \begin{matrix}-2\\2\end{matrix} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3/2 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \Rightarrow x_1 &= d_1 = -2 \\ x_3 &= d_2 = 3/2 \end{aligned}\right) \Rightarrow \underline{x}_{\text{pivot}} \text{ comes from } \underline{d}$$

Combine with nullspace

$$\underline{x}_{\text{complete}} = \underline{x}_p + \underline{x}_n$$

one particular sol. ($A\underline{x}_p = \underline{b}$)

many sol. (a generic vector in $N(A)$) ($A\underline{x}_n = \underline{0}$: comb of $n-r$ special sol.)

Recall: special sol.s to $A\underline{x}_n = \underline{0}$

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ complete sol. to $A\underline{x} = \underline{b} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

$$\underline{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

($N(A)$ is a 2D subspace of R^4

$\Rightarrow$ complete sol. forms a plane parallel to $N(A)$ and passes through

$\underline{x}_p = (-2, 0, 3/2, 0)$)

Q: If A is square, invertible, what are $\underline{x}_p$ & $\underline{x}_n$? $(m=n=r)$

$\underline{x}_p = A^{-1}\underline{b}$ (the only sol.)

# of free var.s $= n-r = 0$

$\Rightarrow$ no special sol.

$\Rightarrow R = I$ has no zero rows

$\Rightarrow N(A)$ contains only $\underline{0}$

$\Rightarrow \underline{x}_{\text{complete}} = A^{-1}\underline{b} + \underline{0} = A^{-1}\underline{b}$

(Situation in ch. 2, $[A\ \underline{b}] \to [I\ A^{-1}\underline{b}]$)

(in general $[R\ \underline{d}]$)

Rank

rank = # of nonzero pivots

If $A_{m\times n}$ is of rank r

$\Rightarrow r \le m,\ r \le n$

Full col. rank $(r=n)$

1. All col.s of A are pivot col.s
2. # of free var.s $= n-r=0$

   (no free var.s)
3. $N(A) = \{\underline{0}\}$
4. $A\underline{x} = \underline{b}$: $\underline{x} = \underline{x}_p$ unique sol. if it exists

   (0 or 1 sol)

Ex:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \to R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A\underline{x} = \underline{b}$ (0 or 1 sol.)

has sol. if $\underline{b} \in C(A)$

Let $\underline{b} = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ only unique sol.

(sum of 2 col.s)

<u>In general</u>

$\therefore r \le m$ & $r = n \Rightarrow n \le m$ ($n < m$: overdetermined)

A is tall & thin & $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

For any $\underline{b} \in R^m$ not a comb. of col.s of $A \Rightarrow$ no sol.

<u>Full row rank</u> ($r = m$)

Can solve $A\underline{x} = \underline{b}$ for every $\underline{b}$

(no zero rows $\Rightarrow$ no constr. on $\underline{b}$)

# of free var.s $= n - r = n - m$

$\Rightarrow n - m$ special sol.s to $A\underline{x} = \underline{0}$

($m \le n$, if $m < n$ underdetermined)

Ex: $x + y + z = 3$ $\quad (r = m = 2)$

$x + 2y - z = 4$

$$[A\ \underline{b}] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & \boxed{\begin{matrix} 3 \\ -2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} \end{bmatrix} = [R\ \underline{d}]$$

$$\underline{s} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \qquad \underline{x}_p = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\underline{x}_{complete} = \underline{x}_p + \underline{x}_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

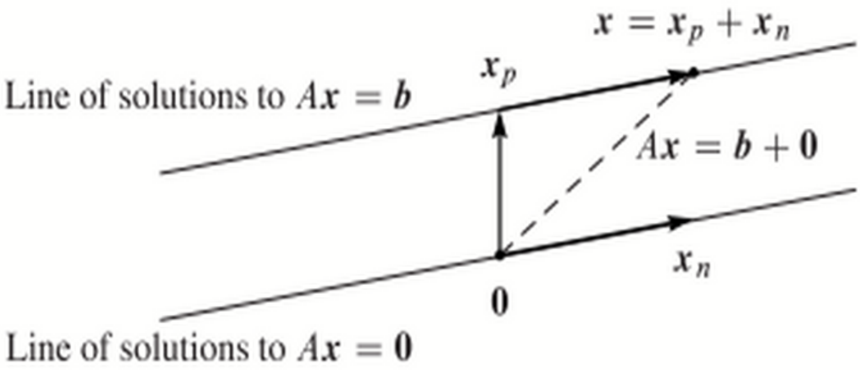

Figure 20: Complete solution = *one* particular solution + *all* nullspace solutions.

In general, if A is of full row rank

1. All rows have pivots, R has no zero rows

2. $A\underline{x} = \underline{b}$ has a sol. for every $\underline{b}$

3. $C(A)$ is the entire R^m

4. There are $n-r = n-m$ special sols to $N(A)$

Full row & col. rank ($r=m=n$)

1. A is invertible & square

2. $R = I$

3. $N(A) = \{\underline{0}\}$

4. $A\underline{x} = \underline{b}$ has a unique sol. for every $\underline{b}$

$$\left(\left. \begin{array}{l} \text{Full col. rank} \Rightarrow \text{uniqueness} \\ \text{Full row rank} \Rightarrow \text{existence} \end{array} \right\} \Rightarrow \text{both} \right)$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \to R = I$

Summary

$r=m=n$	$r=n<m$	$r=m<n$	$r<m, r<n$
$R = I$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$R = \begin{bmatrix} I & F \end{bmatrix}$	$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
one sol. to $A\underline{x} = \underline{b}$	(0 or 1 sol.)	infinitely many	0 or infinitely many
square & invertible	tall & thin	short & wide	Not full rank