Null space of A

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$(N(A))$

The nullspace of A is the collection of all sol.s $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to the eqn $A\underline{x} = \underline{0}$

Recall: $C(A)$ is a subspace of R^4

But $N(A)$ is a subspace of R^3

In general,

$\boxed{\text{Def}}$ $A_{m\times n}$, the nullspace of A, $N(A)$, consists of all sol.s to $A\underline{x} = \underline{0}$

$\underline{x} \in R^n \Rightarrow N(A)$ is a subspace of R^n

Q: Is $N(A)$ a subspace?

If $\underline{x}_1, \underline{x}_2 \in N(A)$

then $A\underline{x}_1 = \underline{0}$, $A\underline{x}_2 = \underline{0}$

$\Rightarrow A(\underline{x}_1 + \underline{x}_2) = A\underline{x}_1 + A\underline{x}_2 = \underline{0} + \underline{0} = \underline{0}$

$A(c\underline{x}) = cA\underline{x} = c\cdot\underline{0} = \underline{0}$

Back to Ex:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because$ col. 1 + col. 2 = col. 3

$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in N(A)$, in fact $N(A)$ is a line through $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ & origin

Other value of $\underline{b}$

sol. to the eqn:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Q: Do the sol. form a subspace

No. $\because$ $\underline{0}$ is NOT a sol. to this eqn

In fact, the set of sol. forms a line in $\mathbb{R}^3$ through $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ but NOT $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Q: How do we compute $N(A)$?

Use elimination (even for singular & rectangular matrices)!

(Elimination does NOT change sol. to $A\underline{x} = \underline{0}$ so nullspace unchanged, but col. space changed)

Solving $A\underline{x}=\underline{0}$: pivot variables, special sols

Recall: nullspace of A is made up of vectors $\underline{x}$ for which $A\underline{x}=\underline{0}$

Ex:

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}$$

(cols of A are NOT lin. indep., col. 2 $= 2\cdot$ col. 1)

(We don't need to use an augmented matrix $\therefore\ \underline{b}=\underline{0}$) (any operation still $\underline{0}$)

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}\rightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}$$

(0 pivot & no row exchange saves it)

(this col. depends on previous col.s)

Q: What to do next?

Move on to next col. in the same row

$$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U$$

echelon form (staircase) (2 pivots)

rank of A = # of pivots = 2 (= # of nonzero rows) ($\because$ each pivot for each nonzero row)

Pivot col.s & free col.s

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

← row 3 (eqn 3) is a lin. comb. of row 1 & 2

free col.s

pivot col.s

⇒ We can assign any number to x_2 & x_4 (free variables)

(free col is comb. of previous col.s)

Special Sol.s ($U\underline{x} = \underline{0}$)

$(x_2, x_4) = (0, 1)$ or $\underline{(1, 0)}$ (set one of free variables to 0)

By back substitution,

$$2x_3 + 4x_4 = 0 \Rightarrow x_3 = 0$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \Rightarrow x_1 = -2$$

⇒ one special sol. $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Similarly, $(x_2, x_4) = (0, 1) \Rightarrow \underline{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Complete sol.

(For $A\underline{x} = \underline{0}$)

special sol. (1, 0)

special sol. (0, 1)

$$\underline{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

(The nullspace contains comb. of all special sol.s)

(# special sol.s = # of free variables
$= n - r = 4 - 2 = 2$)

# of pivots = rank of A

$\boxed{\text{Fact}}$ Suppose $A\underline{x} = \underline{0}$ has more unknowns than eqns ($n > m$, more col.s than rows)

$\Rightarrow \exists$ free col.s

(At least $n - m$ free variables)

($\because$ # of pivots $\leq m$)

Note: when there is free variable it can be set to 1

$\Rightarrow \exists$ nonzero $\underline{x}$ s.t. $A\underline{x} = \underline{0}$

(infinite # of sol.s since $c\underline{x}$ also a sol.)

($N(A)$ contains at least a line if more free variables $N(A)$ larger)

$\boxed{\text{Fact}}$ Dimension of $N(A)$ = # of free variables

Reduced row echelon form

$R =$ reduced row echelon form (rref)

(zeros below & above pivots = 1)

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(scale to 1)

$$\rightarrow \begin{bmatrix} \boxed{\begin{matrix}1\\0\end{matrix}} & \begin{matrix}2\\0\end{matrix} & \boxed{\begin{matrix}0\\1\end{matrix}} & \begin{matrix}-2\\2\end{matrix} \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot cols

$$A\underline{x} = \underline{0} \rightarrow U\underline{x} = \underline{0} \rightarrow R\underline{x} = \underline{0}$$

<u>With proper col. change</u> $(N(A) = N(U)$
$= N(R))$

$$I \leftarrow \begin{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{pivot col.s}} & \underbrace{\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}}_{\text{free col.s}} \\ 0 \quad 0 & 0 \quad 0 \end{bmatrix} \rightarrow F$$

Recall:

$$\underline{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix} = -F$$

(obtain special sol. too?)

<u>Q: why?</u>

Let N contains col.s of special sol.

(nullspace matrix)

$$R = \left[\begin{array}{cc} I_{r\times r} & F \\ 0 & 0 \end{array}\right] \text{) pivot rows}$$

$$\underbrace{\quad}_{\text{pivot col.s}} \quad \underbrace{\quad}_{n-r \text{ free col.s}}$$

Since N contains col.s of special sol. $\underline{s}$

s.t. $R\underline{s} = \underline{0}$ $\Rightarrow$ $RN = 0_{m\times(n-r)}$

Let $N = \begin{bmatrix} -F \\ I_{(n-r)\times(n-r)} \end{bmatrix}$

$$\Rightarrow RN = 0$$

(So special sol. has the form $-F$)

Back substitution on $R\underline{x} = \underline{0}$

$$\Rightarrow \begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} \underline{x}_{pivot} \\ \underline{x}_{free} \end{bmatrix} = \underline{0}$$

$$\Rightarrow I\underline{x}_{pivot} = -F\underline{x}_{free}$$

In each special sol. $\underline{x}_{free}$ is a col. of

$I \Rightarrow \underline{x}_{pivot}$ is a col. of $-F$

$$\left(N = \begin{bmatrix} -F \\ I \end{bmatrix} \right)$$

More on pivot & free col.s

$\boxed{\text{Fact}}$ The # of the pivot col. of A is the same as R & pivot col.s of A is the first r col.s of E^{-1} where $EA = R$

$$A = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\1\end{matrix}} & \begin{matrix}3\\0\\3\end{matrix} & \boxed{\begin{matrix}0\\1\\1\end{matrix}} & \begin{matrix}2\\4\\6\end{matrix} & \begin{matrix}-1\\-3\\-4\end{matrix} \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 1 & 3 & 1 & 6 & -4 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix} \to \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \begin{matrix}3\\0\\0\end{matrix} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \begin{matrix}2\\4\\0\end{matrix} & \begin{matrix}-1\\-3\\0\end{matrix} \end{bmatrix} = R$$

$$\left(P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_{21} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \right)$$

$$E = E_{32} E_{21} P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$E^{-1} = P_{23}^{-1} E_{21}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix}1\\0\\1\end{matrix}} & \boxed{\begin{matrix}0\\1\\1\end{matrix}} & \begin{matrix}0\\1\\0\end{matrix} \end{bmatrix}$$

pivot col.s of A

($EA = R \Rightarrow A = E^{-1}R$
I in R picks out the first two col.s
of E^{-1} to form the corr. col.s of A)

$\boxed{\text{Fact}}$ The pivot cols are NOT comb. of earlier col.s. But the free col.s are comb. of earlier col.s & the comb.s are given by special sol.s ▽

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \to \begin{array}{c} \begin{array}{ccccc} & F & & F & \end{array} \\ \begin{bmatrix} 1 & \boxed{\begin{matrix} 3 \\ 0 \end{matrix}} & 0 & \boxed{\begin{matrix} 2 & -1 \\ 4 & -3 \end{matrix}} \\ 0 & 0 & 0 & 0 \quad 0 \end{bmatrix} \end{array} = R$$

For R, Col 2 = 3 · col 1 + 0 · col 3
Col 4 = 2 · col 1 + 4 · col 3
Col 5 = −1 · col 1 − 3 · col 3

⇒ −3 · col 1 + 0 · col 3 + 1 · col 2 = 0
⇒ −2 · col 1 − 4 · col 3 + 1 · col 4 = 0
⇒ +1 · col 1 + 3 · col 3 + 1 · col 5 = 0

Same for A since $A\underline{x} = \underline{0}$ exactly when $R\underline{x} = \underline{0}$