EE 203001 SES # 1

Introduction to vectors

Vectors & Linear Combinations

Q: Why do we need vectors?

We cannot add apples & oranges

A column vector

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \begin{matrix} \leftarrow \text{first component} \\ \leftarrow \text{second component} \end{matrix}$$

Vector addition

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \underline{v} + \underline{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

Note

$$\underline{v} + \underline{w} = \underline{w} + \underline{v}$$

Scalar Multiplication

$$c\,\underline{v} = \begin{bmatrix} c v_1 \\ c v_2 \end{bmatrix}$$

Ex

$$2\underline{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}, \quad -\underline{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$$

Linear Combination

$$c\underline{v} + d\underline{w}$$

Special cases

$1\underline{v} + 1\underline{w}$ = sum of vectors

$1\underline{v} - 1\underline{w}$ = difference of vectors

$0\underline{v} + 0\underline{w}$ = zero vector

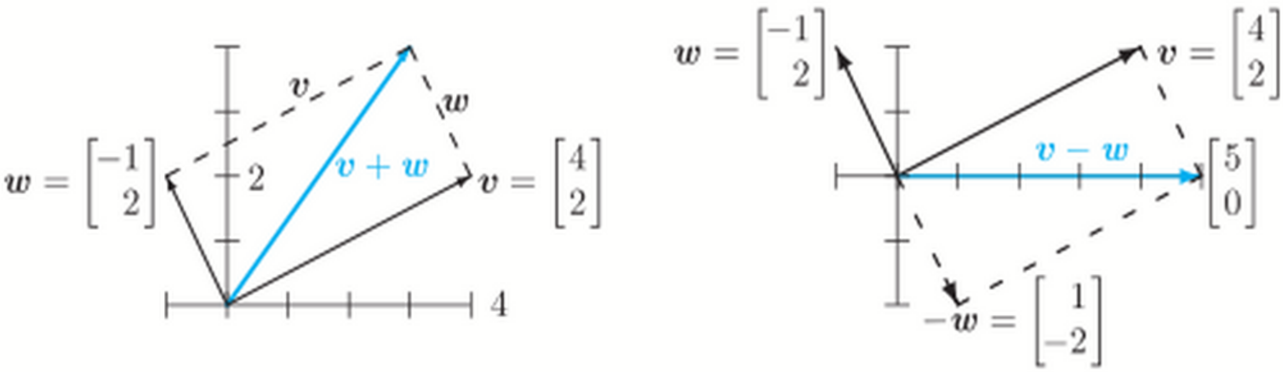
$$\left[\ \underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq 0\ \right]$$

$c\underline{v} + 0\underline{w}$ = vector $c\underline{v}$ in the direction of $\underline{v}$

Q: How to represent vector $\underline{v}$?

Point in the plane

Two numbers $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ + Arrow from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$



$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Figure 1: Vector addition $\boldsymbol{v} + \boldsymbol{w} = (3,\ 4)$ produces the diagonal of a parallelogram. The linear combination on the right is $\boldsymbol{v} - \boldsymbol{w} = (5,\ 0)$.

Vectors in 3-Dim

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \begin{array}{l} \leftarrow \text{ 1st component} \\ \leftarrow \text{ 2nd } \quad '' \\ \leftarrow \text{ 3rd } \quad '' \end{array}$$

Visualization

column vector $\underline{v}$ $\longleftrightarrow$ arrow from origin + pts where arrow ends

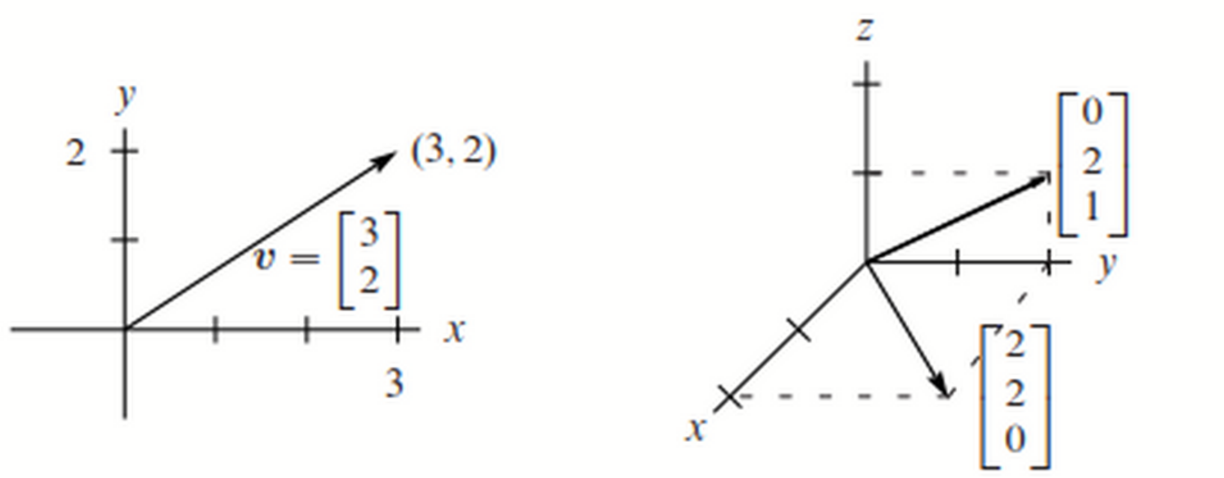


Figure 2: Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z).

Notation

$\underline{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ same as $\underline{v} = (1, 1, -1)$ (save space)

diff. from $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$

(column vector) (row vector)

$\left(\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^T \right)$

Vector Addition

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad \underline{v} + \underline{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$$

Linear Combination

$$c\underline{u} + d\underline{v} + e\underline{w}$$

Ex

$$\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}$$

Q: What's the picture of all combinations of $c\underline{u}$?

The combinations of $c\underline{u}$ fill a line

(exception: if $\underline{u} = \underline{0}$)

Q: What's the picture of all combinations of $c\underline{u} + d\underline{v}$?

The combinations of $c\underline{u} + d\underline{v}$ fill a plane

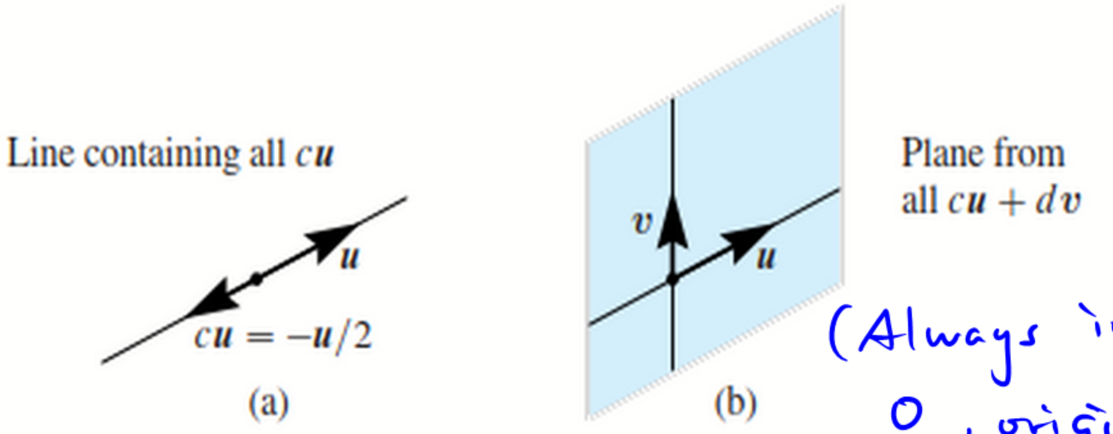
(exceptions: if $\underline{u} = \underline{v} = \underline{0}$

or

$\underline{u}$ & $\underline{v}$ in the same direction)

Q: What's the picture of all combinations of $c\underline{u} + d\underline{v} + e\underline{w}$?

The combinations of $c\underline{u} + d\underline{v} + e\underline{w}$ fill 3-D space



(Always include $\underline{0}$, origin)

Figure 3: (a) Line through $\boldsymbol{u}$. (b) The plane containing the lines through $\boldsymbol{u}$ and $\boldsymbol{v}$.

(exceptions: $\underline{u} = \underline{v} = \underline{w} = \underline{0}$

or

$\underline{w}$ lies on the plane of $c\underline{u} + d\underline{v}$)

Q: How about n-dim. vectors

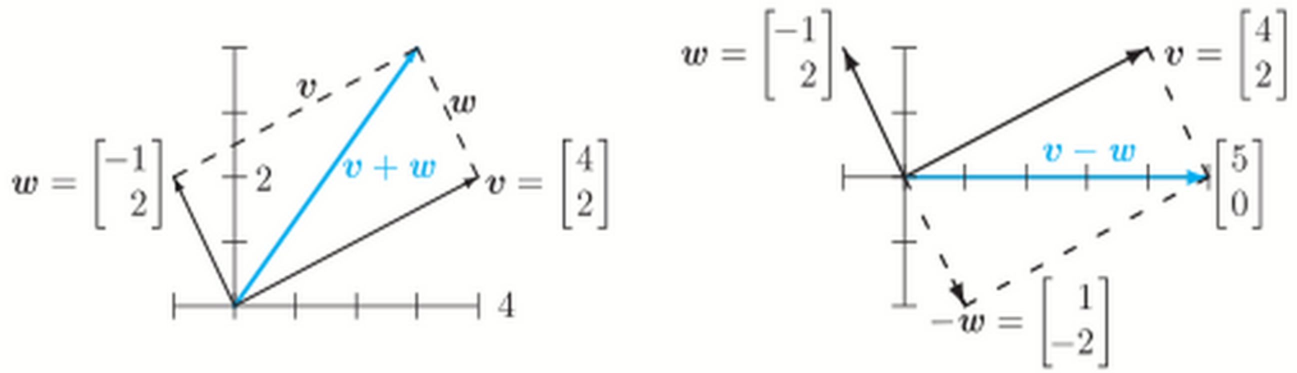
Can be easily generalized from the above concepts!

$$\underline{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \quad \cdots$$

(For ease of visualization, focus on 2 or 3-D vectors as examples)

Length & Dot Products

Def The dot product or inner product of $\underline{v} = (v_1, v_2)$ & $\underline{w} = (w_1, w_2)$ is $\underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2$



$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Figure 1: Vector addition $\boldsymbol{v} + \boldsymbol{w} = (3,\ 4)$ produces the diagonal of a parallelogram. The linear combination on the right is $\boldsymbol{v} - \boldsymbol{w} = (5,\ 0)$.

Q: What is the dot product of $\underline{v}$ & $\underline{w}$ in Fig 1?

$$\underline{v} \cdot \underline{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0$$

(Two vectors are perpendicular!)

(Another easy example $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$
$\qquad \underline{\hat{\imath}} \qquad \underline{\hat{\jmath}}$

)

Note $\underline{w} \cdot \underline{v} = \underline{v} \cdot \underline{w}$

For general n-dim vectors

$$\underline{v} \cdot \underline{w} = \sum_{i=1}^{n} v_i w_i$$

Length & Unit Vectors

$\boxed{\text{Def}}$ The length of a vector $\underline{v}$ is the square root of $\underline{v} \cdot \underline{v}$:

$$\text{length} = \|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}$$

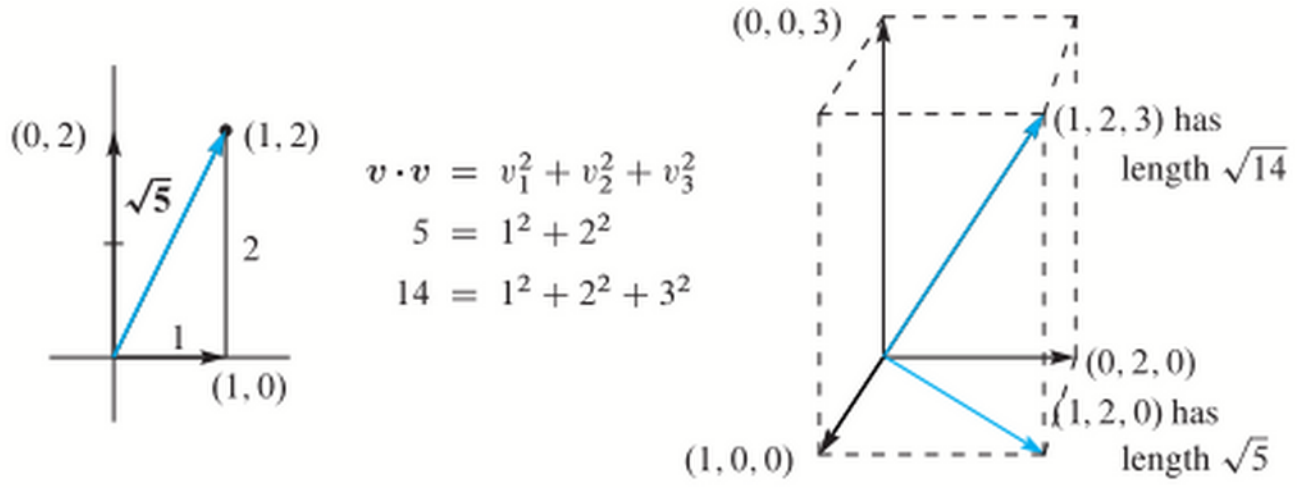


Figure 6: The length $\sqrt{v \cdot v}$ of two-dimensional and three-dimensional vectors.

$\boxed{\text{Def}}$ A unit vector is a vector of length $= 1$, i.e., $\underline{u} \cdot \underline{u} = 1$

Q: How to get unit vector?

$$\underline{u} = \frac{\underline{v}}{\|\underline{v}\|} : \quad \frac{(1,1,1,1)}{\sqrt{4}} = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

(unit vector of same direction as $\underline{v}$)

Standard unit vectors

$$\underline{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\left(\sqrt{\underline{u} \cdot \underline{u}} = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \right)$$

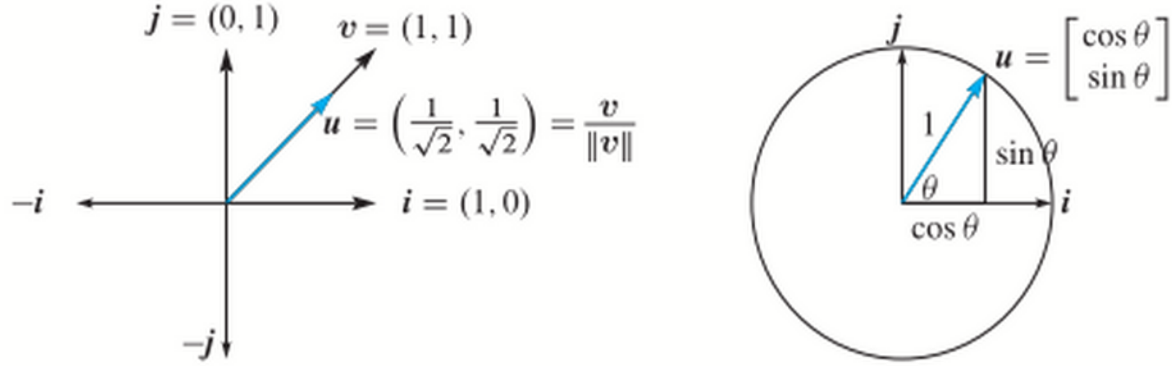
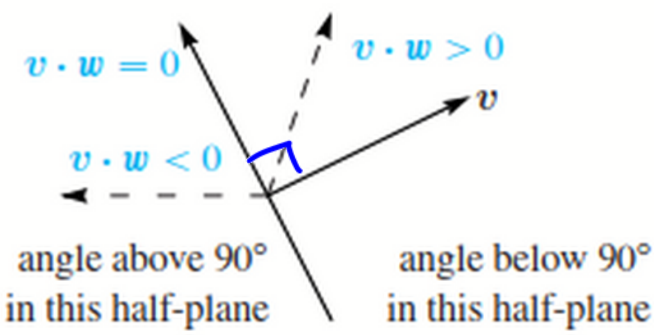


Figure 7: The coordinate vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. The unit vector $\boldsymbol{u}$ at angle 45° (left) divides $\boldsymbol{v} = (1,1)$ by its length $\|\boldsymbol{v}\| = \sqrt{2}$. The unit vector $\boldsymbol{u} = (\cos\theta, \sin\theta)$ is at angle θ.

$\left(\theta = 0 \Rightarrow \underline{u} = \underline{i}, \quad \theta = 90^\circ \text{ or } \frac{\pi}{2} \text{ radians} \Rightarrow \underline{u} = \underline{j} \right)$

The Angle between two vectors



$\boxed{\text{Fact}}$ Unit vectors $\underline{u}_1$ & $\underline{u}_2$ with angle θ in between have

$$\underline{u}_1 \cdot \underline{u}_2 = \cos\theta$$

$$\Rightarrow |\underline{u}_1 \cdot \underline{u}_2| \leq 1$$

$\cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \cos(\beta - \alpha)$

$= \cos\theta$

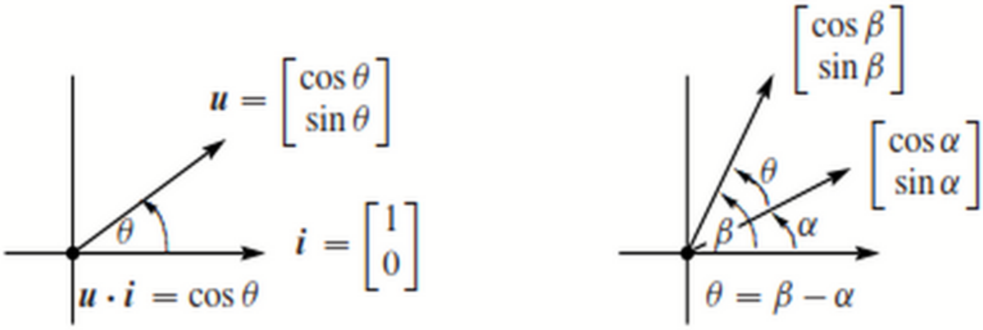


Figure 9: The dot product of unit vectors is the cosine of the angle θ.

$\boxed{\text{Fact}}$ Cosine formula

If $\underline{v}$ & $\underline{w}$ are nonzero vectors

then $\dfrac{\underline{v} \cdot \underline{w}}{\|\underline{v}\|\,\|\underline{w}\|} = \cos\theta$

($\dfrac{\underline{v}}{\|\underline{v}\|}$, $\dfrac{\underline{w}}{\|\underline{w}\|}$ are unit vectors)

$\boxed{\text{Fact}}$ Schwarz inequality

$|\underline{v} \cdot \underline{w}| \le \|\underline{v}\|\,\|\underline{w}\|$

(comes from $|\cos\theta| \le 1$)

$\boxed{\text{Fact}}$ Triangle inequality

$\|\underline{v} + \underline{w}\| \le \|\underline{v}\| + \|\underline{w}\|$