EE 203001 SBS #35

Change of basis & Image compression

Computations can be made easier by an appropriate choice of basis

Applications:

video, music, images are data containing lots of info $\xrightarrow[\text{change of basis}]{}$ efficiently stored & transmitted

Change of basis

Focus on a special lin. transf.

$T(\underline{v}) = \underline{v}$ (identity transf.)

Q: What is the corr. matrix A?

Input basis: $\underline{v_1}, \cdots, \underline{v_n}$

Output basis: $\underline{v_1}, \cdots, \underline{v_n}$

same

Then

$$\left.\begin{aligned} T(\underline{v_1}) &= \underline{v_1} \\ T(\underline{v_2}) &= \underline{v_2} \\ &\vdots \\ T(\underline{v_n}) &= \underline{v_n} \end{aligned}\right) \Rightarrow A = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} = I$$

(same input-output basis $A = I$)

Q: How about diff. input output basis?

Input basis: $\underline{v_1}, \cdots, \underline{v_n}$

Output basis: $\underline{w_1}, \cdots, \underline{w_n}$ } diff.

Then, $T(\underline{v_1}) = m_{11}\underline{w_1} + \cdots + m_{n1}\underline{w_n}$

$\vdots$

$T(\underline{v_n}) = m_{n1}\underline{w_1} + \cdots + m_{nn}\underline{w_n}$

$\Rightarrow A = M$ (change of basis matrix)

Ex 9: (p. 390)

$\underline{v_1} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ $\underline{v_2} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\underline{w_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{w_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Q: What is the change of basis matrix M?

$$\left.\begin{aligned} \underline{v_1} &= 3\underline{w_1} + 7\underline{w_2} \\ \underline{v_2} &= 2\underline{w_1} + 5\underline{w_2} \end{aligned}\right\} \Rightarrow M = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

Note:

Input basis: $\underline{v_1}, \underline{v_2}$

output basis: $\underline{w_1}, \underline{w_2}$ (standard basis)

$$\underbrace{\begin{bmatrix} 3 \\ 7 \end{bmatrix}}_{\underline{v_1} \text{ in standard basis (or basis of } \underline{w}\text{'s)}} = M \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{v_1} \text{ (in basis of } \underline{v}\text{'s)}}$$

Ex 10:

$$\underline{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \underline{w_1} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \quad \underline{w_2} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\underline{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \underline{w_1} + c_2 \underline{w_2} = c_1 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow c_1 = 5, \quad c_2 = -7$$

$$\underline{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = d_1 \underline{w_1} + d_2 \underline{w_2} = d_1 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + d_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow d_1 = -2, \quad d_2 = 3$$

$$\text{So } M' = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = M^{-1}$$

$$\left(\begin{bmatrix} \underline{v_1} & \underline{v_2} \end{bmatrix} = \begin{bmatrix} \underline{w_1} & \underline{w_2} \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \right.$$

$$\left. \Rightarrow \quad I = M M^{-1} \right)$$

Note:

In general, we have

$$\begin{bmatrix} \underline{v_1} & \cdots & \underline{v_n} \end{bmatrix} = \begin{bmatrix} \underline{w_1} & \cdots & \underline{w_n} \end{bmatrix} M$$

$$V = W \quad M$$

(from basis of $\underline{v}$'s to standard basis)	(from basis of $\underline{w}$'s to standard basis)	(from basis of $\underline{v}$'s to basis of $\underline{w}$'s)
$\underline{v} \to \underline{e}$	$\underline{w} \to \underline{e}$	$\underline{v} \to \underline{w}$

Image compression

Change of basis:

Standard $\longrightarrow$ wavelet or Fourier basis

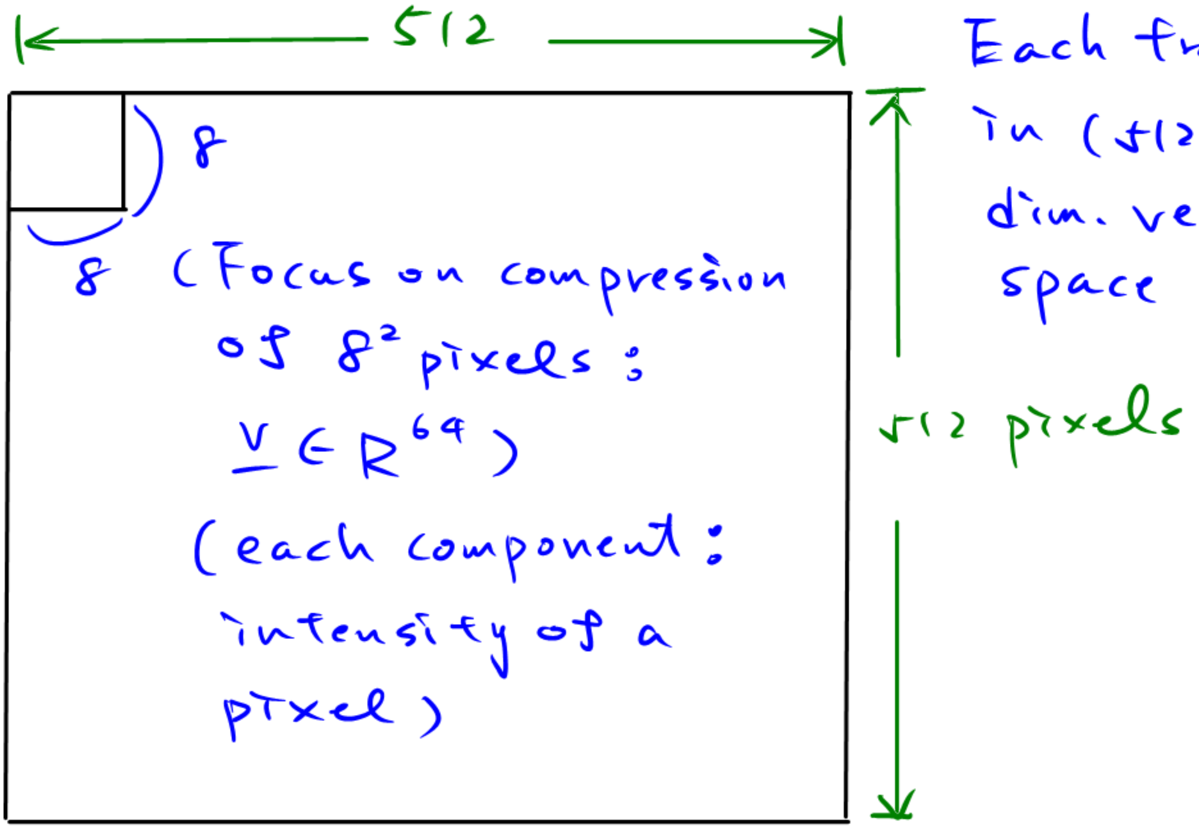
$$V = WM \Rightarrow I = WM \Rightarrow M = W^{-1}$$

$$\Rightarrow \underline{c} = W^{-1}\underline{v} \quad \text{or} \quad \underline{v} = W\underline{c}$$

Alternatively,

$$\begin{aligned}\underline{v} &= c_1\underline{w}_1 + \cdots + c_n\underline{w}_n \\ &= [\underline{w}_1 \cdots \underline{w}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= W\underline{c}\end{aligned}$$

Idea of image compression:



Each frame in $(512)^2$ dim. vector space

Compression:

Signal too long, want to compress it

Say, keeping only largest 5% of coeff.s

$\Rightarrow$ 20 : 1 compression

Q: Why change of basis?

If keep only 5% of standard basis, 95% of images disappear!

If we choose a better basis

5% of basis vectors may come very close to the original image!

(Basically, want many small c's in the new coord.) (close to 0)

Image compression:

input $\underline{v}$ $\longrightarrow$ coeff.s $\underline{c}$ $\longrightarrow$ compressed $\underline{\hat{c}}$ $\longrightarrow$ compressed $\underline{\hat{v}}$

(lossless) (lossy) (reconstruct)

(Want to find good basis s.t. loss is small $\underline{c} \to \underline{\hat{c}}$)

Note: In vedio, not only compress each frame, we only need to encode & compress difference of each frame

(Little diff. from frame to frame)

Wavelet basis

Haar wavelets basis: $(n=4)$

(orthogonal basis)

$$\underline{w_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{w_2} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad \underline{w_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{w_4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

(average) (localize in 1st half) (localize in 2nd half)

(Half zeros / Half ones except $\underline{w_1}$)

Q: Why is this called a wavelet?

See this in plot!

Q: How to do compression?

remove c's below a threshold

Ex: $\underline{v} = (6,4,5,1)$

$$\underline{v} = W\underline{c} \Rightarrow \begin{bmatrix} 6 \\ 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$\Rightarrow \underline{c} = W^{-1}\underline{v}$$

Find W^{-1}:

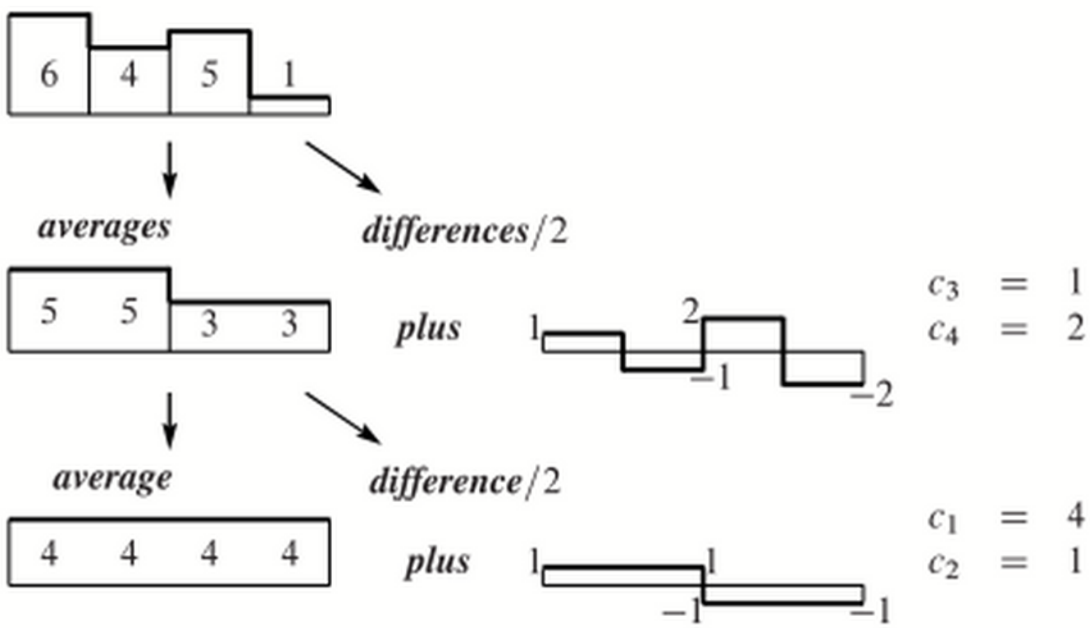
$$\left(W \begin{bmatrix} \frac{1}{\sqrt{4}} & & & \\ & \frac{1}{\sqrt{4}} & & \\ & & \frac{1}{\sqrt{2}} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix} \right)^{-1} = \begin{bmatrix} \frac{1}{\sqrt{4}} & & & \\ & \frac{1}{\sqrt{4}} & & \\ & & \frac{1}{\sqrt{2}} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix} W^T$$

$$\Rightarrow \begin{bmatrix} \ddots \end{bmatrix}^{-1} W^{-1} = \begin{bmatrix} \ddots \end{bmatrix} W^T$$

$$\Rightarrow W^{-1} = \begin{bmatrix} \ddots \end{bmatrix}^2 W^T = \begin{bmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

So $$\underline{c} = W^{-1}\underline{v} = \begin{bmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 16 \\ 4 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Faster "multiscale" method



(JPEG 2000 improves on Haar wavelets)

Fourier basis (DFT, $n = 4$)

$$\omega = e^{i 2\pi/4} = e^{i\pi/2} = i$$

$$\begin{array}{c} F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{w_1} \quad \underline{w_2} \quad \underline{w_3} \quad \underline{w_4} \end{array}$$

Change of basis:

$$\underline{v} = W\underline{c} = F\underline{c}$$ (speed up by FFT)

$$\Rightarrow \underline{c} = F^{-1}\underline{v}$$

Find F^{-1}:

$$F^{-1} = \frac{1}{4}\overline{F} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix}$$

$$\Rightarrow \underline{c} = F^{-1}\underline{v} = \frac{1}{4}\overline{F}\,\underline{v}$$

(speed up by FFT)

(basis used by JPEG are cosines — real part of e^{ik})