EE203001 SES #34

Linear transformations & their matrices

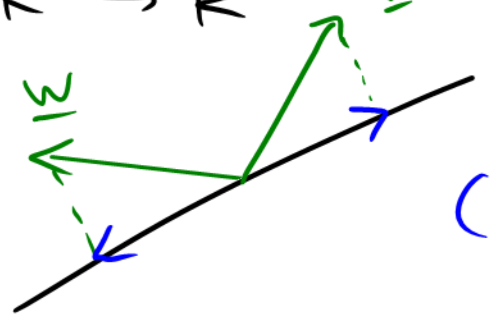
Two approaches:
- No coord. (No matrix) (geometric approach)
- With coord. (matrix?)

Without coord. (No matrix)

Ex 1: Projection

Describe projection as a lin. transformation

$T : \mathbb{R}^2 \to \mathbb{R}^2$



$\boxed{\text{Def}}$ A transformation T is linear if

$$T(\underline{v}+\underline{w}) = T(\underline{v}) + T(\underline{w})$$

$$T(c\underline{v}) = cT(\underline{v})$$

$\forall\ \underline{v}\ \&\ \underline{w}\quad \forall$ scalar c

Equivalently,

$$T(c\underline{v}+d\underline{w}) = cT(\underline{v}) + dT(\underline{w})$$

$\forall\ \underline{v}\ \&\ \underline{w}\quad \forall\ c\ \&\ d$

Note: $T(\underline{0}) = \underline{0}$ $\quad (T(c\underline{0}) = cT(\underline{0}))$

Non-ex 1: Shift the whole plane

Consider $T(\underline{v}) = \underline{v} + \underline{v_0}$

(Shift every vector in the plane by adding a fixed vector $\underline{v_0}$ to it)

NOT a lin. transf.!

since $T(2\underline{v}) = 2\underline{v} + \underline{v_0} \neq 2T(\underline{v})$

Non-ex 2:

Consider $T(\underline{v}) = \|\underline{v}\|$

(take any vector to its length)

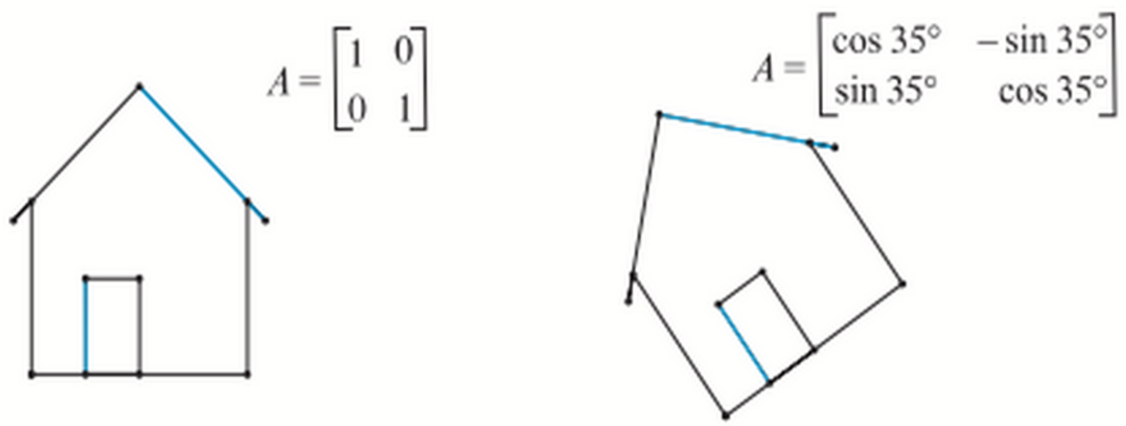
NOT a lin. transf.!

Since $T(c\underline{v}) = |c|\,\|\underline{v}\| \neq c\,T(\underline{v})$ if $c < 0$

Focus on lin. transf.

Ex 2: Rotation by $35°$

$T: \mathbb{R}^2 \to \mathbb{R}^2$



(Don't need coord.)

The big picture

Geometric approach (no coord.) $\xleftrightarrow{\text{v.s.}}$ matrix approach (coord.)

Help us see big picture !

(rotation of house)

More detailed descriptions !

With coord. (matrix !)

All lin. transf. described above can be described in terms of matrices !

In fact, lin. transf. are abstract description of mul. by a matrix !

Ex 3: $T(\underline{v}) = A\underline{v}$

Q: Is this indeed a lin. transf. ?

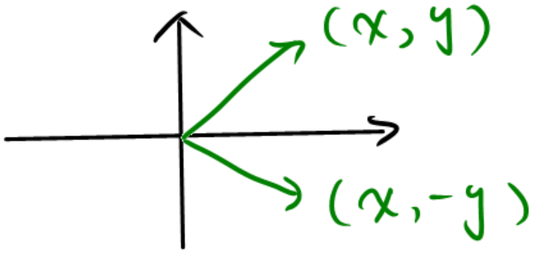
$$T(\underline{v}+\underline{w}) = A(\underline{v}+\underline{w}) = A\underline{v} + A\underline{w}$$
$$= T(\underline{v}) + T(\underline{w}) \ (\checkmark)$$

$$T(c\underline{v}) = A(c\underline{v}) = cA\underline{v} = cT(\underline{v}) \ (\checkmark)$$

Ex 4: Suppose $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Q: How do we describe $T(\underline{v}) = A\underline{v}$ geometrically ?

$$A\underline{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \begin{matrix} \leftarrow \text{unchanged} \\ \leftarrow \text{minus sign} \end{matrix}$$



$T(\underline{v}) = A\underline{v}$: reflects xy-plane across the x-axis

Ex 5: $T: \mathbb{R}^3 \to \mathbb{R}^2$

Q: How do we find T that takes 3D space to 2D space?

Any 2×3 matrix A & $T(\underline{v}) = A\underline{v}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Describing $T(\underline{v})$

Q: How much info. do we need to determine $T(\underline{v})$ $\forall$ $\underline{v}$?

If we know $T(\underline{v}_1)$,

we know $T(c\underline{v}_1) = cT(\underline{v}_1)$

If we know $T(\underline{v}_1)$ & $T(\underline{v}_2)$ for indep. $\underline{v}_1$ & $\underline{v}_2$

we know $T(c\underline{v}_1 + d\underline{v}_2)$

$$= cT(\underline{v}_1) + dT(\underline{v}_2)$$

(we can predict how T transform any vector in the space spanned by $\underline{v}_1$ & $\underline{v}_2$)

If we want to know $T(\underline{v})\ \forall\ \underline{v} \in R^n$

Just need to know $T(\underline{v_1}), T(\underline{v_2}) \cdots T(\underline{v_n})$

for any basis of the input space !

Since

$$\underline{v} = c_1 \underline{v_1} + c_2 \underline{v_2} + \cdots + c_n \underline{v_n}$$

(Any $\underline{v}$ can be described as a lin. comb. of basis of R^n)

$\Rightarrow T(\underline{v}) = c_1 T(\underline{v_1}) + c_2 T(\underline{v_2}) + \cdots + c_n T(\underline{v_n})$

(T is a lin. transf.)

Note: This is how we get from a (coord. free) lin. transf. to a (coord. based) matrix (C_i's)

(Every $\underline{v}$ can be written as a lin comb. of basis in exactly one way)

(The coeff.s of these vectors are coord.s)

Note: Coord. comes from basis

(Changing basis $\Rightarrow$ changing coord.)

(standard basis v.s. other basis)

(basis of eigenvectors)

Ex:
$$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix of a lin. transf.

Q: Given a lin. transf. T, how do we find a representing matrix A?

$T : R^n \to R^m$

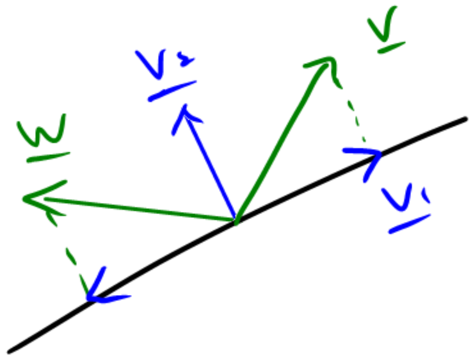
Basis for input vector:

$\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n$ (coord. to input vector)

Basis for output vector:

$\underline{w}_1, \underline{w}_2, \cdots, \underline{w}_m$ (coord. to output vector)

Ex: Projection $(n = m = 2)$



choose $\underline{v}_1$ along the line of projection, $\underline{v}_2$ orthogonal to line of projection

Then, $T(c_1\underline{v}_1 + c_2\underline{v}_2) = c_1\underline{v}_1 + \underline{0}$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A\underline{v} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

($\underline{v}_1 = \underline{w}_1$, $\underline{v}_2 = \underline{w}_2$ same basis for input & output)

(basis are eigenvectors, A becomes diagonal Λ)

Q: What happens if we choose standard basis instead?

Back to example: (say projection onto 45° line)

$\underline{w}_1 = \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{w}_2 = \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (standard basis)

projection matrix $P = \dfrac{\underline{a}^T \underline{a}}{\underline{a}\,\underline{a}^T} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

($\underline{a} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$) (more difficult than basis of eigenvectors)

In general,

$T : R^n \to R^m$

Basis for input vector:

$\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n$ (coord. to input vector)

Basis for output vector:

$\underline{w}_1, \underline{w}_2, \cdots, \underline{w}_m$ (coord. to output vector)

If $T(\underline{v}_1) = a_{11}\underline{w}_1 + a_{21}\underline{w}_2 + \cdots + a_{m1}\underline{w}_m$

then 1st col. of $A = (a_{11}, a_{21}, \cdots, a_{m1})$

If $T(\underline{v}_i) = a_{1i}\underline{w}_1 + a_{2i}\underline{w}_2 + \cdots + a_{mi}\underline{w}_m$

then ith col. of $A = (a_{1i}, a_{2i}, \cdots, a_{mi})$

Ex 6: $T = d/dx$

Let T be a transf. that takes derivatives:

$$T(c_1 + c_2 x + c_3 x^2) = c_2 + 2c_3 x$$

Input space: 3D space of quadratic poly.s $c_1 + c_2 x + c_3 x^2$ with basis of

$\underline{v_1} = 1$, $\underline{v_2} = x$, $\underline{v_3} = x^2$

Output space: 2D space of basis

$\underline{w_1} = \underline{v_1}$, $\underline{w_2} = \underline{v_2}$

(This is lin.!) (chk by det.)

Find A:

$$\left.\begin{aligned} T(\underline{v_1}) &= 0 = 0\underline{w_1} + 0\underline{w_2} \\ T(\underline{v_2}) &= 1 = 1\underline{w_1} \\ T(\underline{v_3}) &= 2x = 2\underline{w_2} \end{aligned}\right) \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}\right) = A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix} \quad (v)$$

Read Ex 4: p. 387

(If bases change, T is the same but A is diff.)

Conclusion

1. For any lin. transf. T, we can find A, s.t. $T(\underline{v}) = A\underline{v}$

2. If the transf. is invertible, the inverse transf. has matrix A^{-1}

3. Product of two transf. $T_1 T_2$

   $T_1 : \underline{v} \to A_1 \underline{v}$, $T_2 : \underline{w} \to A_2 \underline{w}$

   has matrix $A_1 A_2$

   (This is where matrix mul. comes from.)

   Read Ex 7 & 8 (p. 389)