EE 203001 SES # 33

Singular value decomposition (SVD)

Recall: $A = S \Lambda S^{-1}$

problems on S:

(1) S is usually not orthogonal
(unless $A^T = A$)

(2) There are NOT always enough indep. eigenvectors
(A may NOT be diagonalizable)

(3) $A\underline{x} = \lambda \underline{x}$ requires that A is square
(NOT possible for rectangular matrices)

Q: Can we have a more general decomp. that solves all these problems?

Yes! SVD!

Price to pay

Need two sets of singular vectors

For any matrix A, we have the final & best decomp.:

$$A = \underbrace{U}_{\text{orthogonal}} \underbrace{\Sigma}_{\text{diagonal}} \underbrace{V^T}_{V \text{ is orthogonal}}$$

Special case: A is PD

$$A = Q \Lambda Q^T, \quad U = V = Q$$

How it works

Two sets of singular vectors

$\{\underline{v}_1, \cdots, \underline{v}_r\}, \{\underline{u}_1, \cdots, \underline{u}_r\}$

$$A\underline{v}_1 = \sigma_1 \underline{u}_1, \quad \cdots \quad, A\underline{v}_r = \sigma_r \underline{u}_r$$

(instead of $A\underline{x} = \lambda \underline{x}$)

Note:

$\{\underline{v}_1, \cdots, \underline{v}_r\}$: orthogonal basis for row space of A, $C(A^T)$

$\{\underline{u}_1, \cdots, \underline{u}_r\}$: orthogonal basis for col. space of A, $C(A)$

$\{\sigma_1, \cdots, \sigma_r\}$: singular values > 0

Note: $\underline{u}_i = A\underline{v}_i$

Can think of A as lin. transformation taking $\underline{v}_i$ in row space into $\underline{u}_i$ in col. space

SVD: Finding orthogonal basis for row space $\xrightarrow{A}$ orthogonal basis for col. space

Note: NOT hard to find orthogonal basis $\{\underline{v}_1, \cdots, \underline{v}_r\}$ in row space

(use Gram-Schmidt !)

But no reason to expect A transforms $\{\underline{v}_1, \cdots, \underline{v}_r\}$ into another orthogonal basis $\{\underline{u}_1, \cdots, \underline{u}_r\}$!

Q: How about $N(A)$ & $N(A^T)$?

zeros on the diagonal of Σ take care of them !

In matrix form

$$A\underline{v}_i = \sigma_i \underline{u}_i \quad , i = 1, \cdots, r$$

$$\Rightarrow \underset{m\times n}{A} \underbrace{\begin{bmatrix} \underline{v}_1 \cdots \underline{v}_r \end{bmatrix}}_{V_r}^{n\times r} = \underbrace{\begin{bmatrix} \underline{u}_1 \cdots \underline{u}_r \end{bmatrix}}_{U_r}^{m\times r} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}}_{\Sigma_r}^{r\times r}$$

$$\Rightarrow A = U_r \Sigma_r V_r^T$$

(reduced form)

$(V_r^T V_r = I \ , \ U_r^T U_r = I)$

Q: How about $N(A)$ & $N(A^T)$?

Add in $(n-r)$ more $\underline{v}$'s : orthonormal bases for $N(A)$

(orthogonal to $\{\underline{v}_1, \cdots, \underline{v}_r\} \in C(A^T)$)

Add in $(m-r)$ more $\underline{u}$'s : orthonormal bases for $N(A^T)$

(orthogonal to $\{\underline{u}_1, \cdots, \underline{u}_r\} \in C(A)$)

Complete form

$$\underset{m\times n}{A} \underbrace{\begin{bmatrix} \underline{v}_1 \cdots \underline{v}_r \cdots \underline{v}_n \end{bmatrix}}_{V}^{n\times n} = \underbrace{\begin{bmatrix} \underline{u}_1 \cdots \underline{u}_r \cdots \underline{u}_m \end{bmatrix}}_{U}^{m\times m} \underbrace{\begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix}}_{\Sigma}^{m\times n}$$

$\Rightarrow \quad A = U \Sigma V^T$

Note: Σ is NOT square but $m \times n$

Σ is Σ_r with $m-r$ new zero rows

& $n-r$ new zero col.s

$(V^T V = I \ , U^T U = I)$

Alternative form

$$A = U \Sigma V^T = \underline{u}_1 \sigma_1 \underline{v}_1^T + \cdots + \underline{u}_r \sigma_r \underline{v}_r^T$$

(same for both reduced & complete form)

read Ex 1 & Ex 2 (p. 364)

Calculations

2×2 full rank A:

Problem: Find orthonormal vectors $\underline{v}_1$ & $\underline{v}_2$ s.t. $A\underline{v}_1$ & $A\underline{v}_2$ are also orthogonal

(NOT easy to find such $\underline{v}$'s)

$$\Rightarrow A [\underline{v}_1 \ \underline{v}_2] = [\sigma_1 \underline{u}_1 \ \sigma_2 \underline{u}_2]$$

$$= [\underline{u}_1 \ \underline{u}_2] \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}$$

$\Rightarrow A = U \Sigma V^T$

Focus on V first:

$$A^TA = (V\Sigma^T U^T)(U\Sigma V^T)$$
$$= V\Sigma^T\Sigma V^T$$
$$= V\begin{bmatrix}\sigma_1^2 & \\ & \sigma_2^2\end{bmatrix}V^T$$

(recall: $A = Q\Lambda Q^T$)

$\Rightarrow$ cols of V are eigenvectors and σ_i^2 are eigenvalues of A^TA

(True for general $m \times n$ A)

Q: How about $\underline{u}$?

$$A\underline{v}_i = \sigma_i \underline{u}_i \Rightarrow \underline{u}_i = \frac{A\underline{v}_i}{\sigma_i}$$

SVD examples

Ex: (p.366) (full rank)

Find SVD for $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

Focus on V first,

Find eigenvectors for A^TA

$$A^TA = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$|A^TA - \lambda I| = (5-\lambda)^2 - 3^2 = (5-\lambda-3)(5-\lambda+3) = (2-\lambda)(8-\lambda)$$

$\lambda_1 = 2$:

$$A^TA - 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \underline{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_2 = 8$:

$$A^TA - 8I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \underline{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find U:

$$A\underline{v}_1 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix} = \sigma_1 \underline{u}_1$$

$$= \sqrt{2}\, \underline{u}_1 \Rightarrow \underline{u}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A\underline{v}_2 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} = \sigma_2 \underline{u}_2$$

$$= 2\sqrt{2}\, \underline{u}_2 \Rightarrow \underline{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{2} & \\ & 2\sqrt{2} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}$$

Interpretation:

A transform the unit circle to an ellipse (true for any invertible 2×2 matrix)

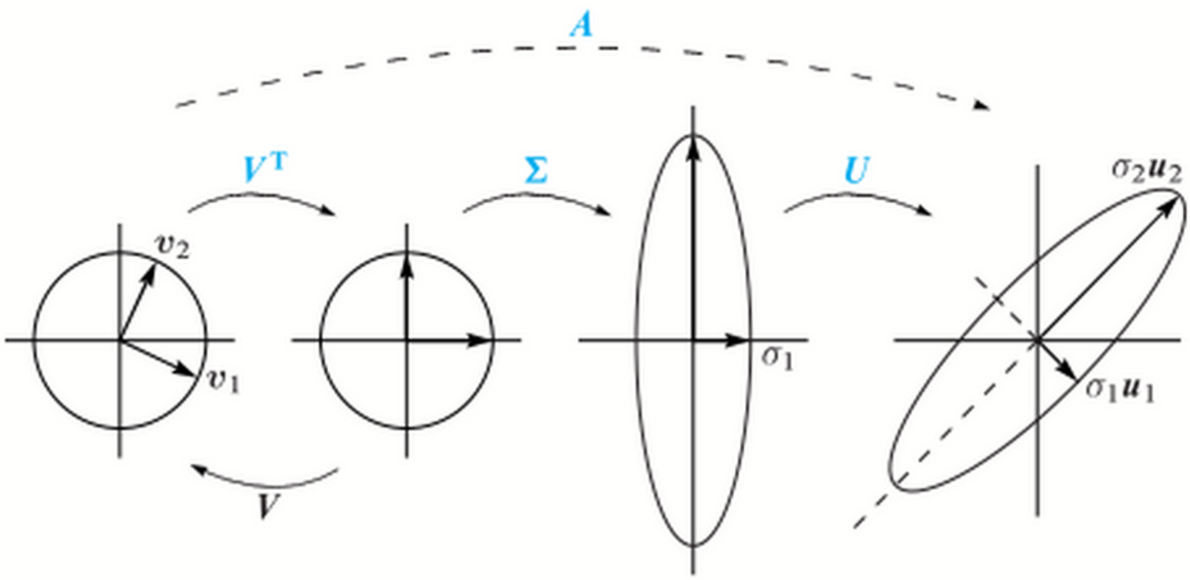

Figure 45: U and V are rotations and reflections. Σ stretches circle to ellipse.

Alternatively, we can find $\underline{u}$'s directly

Reason:

$$AA^T = (U\Sigma V^T)(V\Sigma^T U^T)$$
$$= U\Sigma\Sigma^T U^T$$

$\Rightarrow$ col.s of U are eigenvectors of AA^T

Back to example:

$$AA^T = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

Same eigenvalues $\lambda_1 = 2$, $\lambda_2 = 8$

$\Rightarrow$ $\underline{x}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \underline{u}_1$, $\underline{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underline{u}_2$

Q: Can we choose $\underline{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$?

No! Have to follow $A\underline{v}_2 = \sigma_2 \underline{u}_2$

Ex: (p. 367) (with nullspace)

Find SVD for singular $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

row space has only one basis

$\Rightarrow \quad \underline{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

col. space has only one basis

$\Rightarrow \quad \underline{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Then $\quad \underbrace{\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{v}_1} = \underbrace{\sigma_1}_{\sigma_1} \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\underline{u}_1}$

$\Rightarrow \sigma_1 = \sqrt{10}$

Find the basis in $N(A)$ & $N(A^T)$

$\underline{v}_2 \in N(A)$, $\underline{u}_2 \in N(A^T)$

$$\underbrace{\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}}_{A} = \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{10} & \\ & 0 \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{V^T}$$

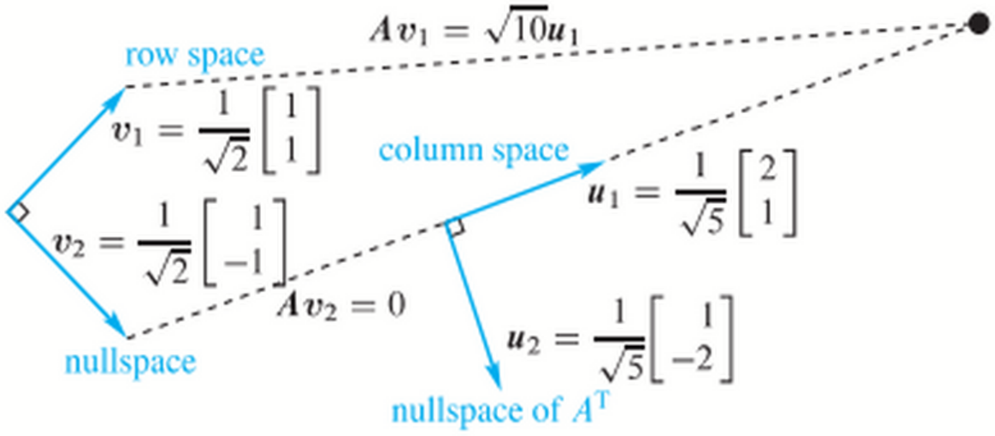

Figure 46: The SVD chooses orthonormal bases for 4 subspaces so that $Av_i = \sigma_i u_i$.

Relation with 4 fundamental subspaces

$\underline{v}_1, \dots, \underline{v}_r$: orthonormal basis for $C(A^T)$
(row space)

$\underline{u}_1, \dots, \underline{u}_r$: orthonormal basis for $C(A)$
(col. space)

$\underline{v}_{r+1}, \dots, \underline{v}_n$: orthonormal basis for $N(A)$
(null space)

$\underline{u}_{r+1}, \dots, \underline{u}_n$: orthonormal basis for $N(A^T)$
(left null space)

Proof of SVD

Let $\underline{v}_i$ be eigenvectors of A^TA

$$\Rightarrow A^TA\underline{v}_i = \sigma_i^2 \underline{v}_i$$

$$\Rightarrow \underline{v}_i^T A^TA\underline{v}_i = \sigma_i^2 \underline{v}_i^T \underline{v}_i$$

$$\Rightarrow \|A\underline{v}_i\|^2 = \sigma_i^2 \|\underline{v}_i\|^2 = \sigma_i^2$$

$$\& \quad AA^TA\underline{v}_i = \sigma_i^2 A\underline{v}_i$$

Let $\underline{u}_i$ be eigenvectors of AA^T

$\Rightarrow \underline{u}_i$ is normalized $A\underline{v}_i$

$$\Rightarrow \underline{u}_i = A\underline{v}_i / \|A\underline{v}_i\| = A\underline{v}_i / \sigma_i$$

$$\Rightarrow A\underline{v}_i = \sigma \underline{u}_i$$