EE203001 SBS #32

More on PD matrices

$\boxed{\text{Fact}}$ If A is PD, A^{-1} is also PD

Reason: If A is PD, A has positive eigenvalues $\lambda_1 > 0, \cdots, \lambda_n > 0$

$\Rightarrow$ A^{-1} has eigenvalues $\lambda_1^{-1} > 0, \cdots, \lambda_n^{-1} > 0$

$\Rightarrow$ A^{-1} is also PD

$\boxed{\text{Fact}}$ If A, B are PD, $A+B$ is also PD

Reason: $\underline{x}^T (A+B) \underline{x} = \underline{x}^T A \underline{x} + \underline{x}^T B \underline{x} > 0$

$\forall\, \underline{x} \neq \underline{0}$ $\Rightarrow$ $A+B$ is PD

If A is $m \times n$ rectangular, A is certainly NOT square & symmetric

But $A^T A$ is square & symmetric!

Q: Is $A^T A$ PD for rectangular $A_{m\times n}$?

Test on energy-based definition!

$\underline{x}^T A^T A \underline{x} = (A\underline{x})^T (A\underline{x}) = \|A\underline{x}\|^2 \geq 0$

only need to chk if $A\underline{x} = \underline{0}$ only when $\underline{x} = \underline{0}$

If A has indep. col., then

$A\underline{x} = \underline{0}$ only when $\underline{x} = \underline{0}$

$\Rightarrow \|A\underline{x}\| > 0 \Rightarrow A^T A$ is PD

Q: Why is this important?

When computing least square sol.

& projection, we work on $A^T A$

$\Rightarrow A^T A$ appears often in applied math!

Another nice feature for PD matrices:

No need to do row exchange for elimination (All pivots > 0)

Similar matrices & Jordan form

Recall: When we have n indep. eigenvectors diagonalization is possible

$\Rightarrow A = S\Lambda S^{-1}$ or $S^{-1}AS = \Lambda$

Q: Can we do sth. similar when diagonalization is NOT possible?

Yes! Similar matrices & Jordan form!

$\boxed{\text{Def}}$ Let M be any invertible matrix

Then $B = M^{-1}AM$ is similar to A

Note: If $B = M^{-1}AM$, then $A = MBM^{-1}$

$\Rightarrow A$ is also similar to B

(simply change M to M^{-1})

Note: For the special case $M = S$

$A = S\Lambda S^{-1}$ & $\Lambda = S^{-1}AS$

$\Rightarrow A$ is similar to Λ

Note: $M^{-1}AM$ appears in change of var.s

Set $\underline{u} = M\underline{v}$

$\frac{d\underline{u}}{dt} = A\underline{u}$ becomes $M\frac{d\underline{v}}{dt} = AM\underline{v}$

$\Rightarrow \frac{d\underline{v}}{dt} = M^{-1}AM\underline{v}$

When $M = S$, $M^{-1}AM = S^{-1}AS = \Lambda$

$\Rightarrow$ differential eqns very easy to solve (maximum in simplicity)

Other choices of M can make the new system triangular (Jordan form) & easier to solve

$\boxed{\text{Fact}}$ (No change in λ's)

Similar matrices A & $M^{-1}AM$ have the same eigenvalues

If $\underline{x}$ is an eigenvector of A

$\Rightarrow M^{-1}\underline{x}$ " " " of $B = M^{-1}AM$

Proof: $A\underline{x} = \lambda \underline{x}$

$$\Rightarrow AMM^{-1}\underline{x} = \lambda \underline{x}$$

$$\Rightarrow M^{-1}AMM^{-1}\underline{x} = \lambda M^{-1}\underline{x}$$

$$\Rightarrow B \underbrace{M^{-1}\underline{x}} = \lambda \underbrace{M^{-1}\underline{x}}$$

(eigenvector for B)

Note: the concept of similar matrices allows us to put matrices into families in which all matrices in the family are similar to each other

$\Rightarrow$ Each family can be represented by an diagonal (or nearly diag.) matrix

Distinct eigenvalues

This indicates that A has a full set of indep. eigenvectors

$\Rightarrow$ A is diagonalizable

$\Rightarrow$ $S^{-1}AS = \Lambda$ $\Rightarrow$ A is similar to Λ

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

so A is similar to Λ

but A is also similar to:

$$\underbrace{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}}_{M^{-1}} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}_{B}$$

& B is similar to Λ

or A is similar to $\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$

In fact, A is similar to all 2×2 matrices with eigenvalues 1 & 3

In general,

If two matrices have same n distinct eigenvalues $\Rightarrow$ they are similar to the same diagonal matrix Λ

Repeated eigenvalues

2x2 Ex: If two eigenvalues of A are the same $\Rightarrow$ may not be possible to diagonalize A

Suppose $\lambda_1 = \lambda_2 = 4$

Family 1: $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ only (two eigenvectors)

Reason: $M^{-1}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M = 4M^{-1}M = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Family 2: $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ (only one eigenvector)

Other family members: (same trace & det)

$$\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}, \begin{bmatrix} a & b \\ (8a-a^2-16)/b & 8-a \end{bmatrix}$$

(None of these matrix are diagonalizable Otherwise, it will be similar to $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$)

Not changed by M	changed by M
Eigenvalues	Eigenvectors
Trace & det	Nullspace
Rank	Col. space
# of indep. eigenvectors	Row space
Jordan form	Left nullspace
	Singular values

Examples of Jordan form

Both $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ & $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ are in Jordan form ("the most diagonal" matrix)

Ex 3: (p. 357)

Jordan matrix J has $\lambda = 5, 5, 5$ on its diagonal with only one eigenvector $(1,0,0)$

$$J = \begin{bmatrix} 5 & 1 & 0 \\ & 5 & 1 \\ & & 5 \end{bmatrix} \text{ then } J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

(rank = 2)

Every similar matrix $B = M^{-1} J M$ has same eigenvalues & rank $(B - 5I) = 2$

$\Rightarrow \dim N(J - 5I) = 3 - 2 = 1$

$\Rightarrow$ only one eigenvector $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ only one eigenvector for B: $M^{-1} x$

Note: J^T has same $\lambda = 5, 5, 5$

& rank $(J^T - 5I) = 2$

$\Rightarrow J^T$ is similar to J

$\Rightarrow J^T = M^{-1} J M$:

$$\begin{bmatrix} 5 & & \\ 1 & 5 & \\ & 1 & 5 \end{bmatrix} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 5 & 1 & \\ & 5 & 1 \\ & & 5 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

only one eigenvector for J^T:

$M^{-1}(1,0,0) = (0,0,1)$

Key fact from Jordan's Thm:

J (Jordan form) is similar to every matrix A with $\lambda = 5, 5, 5$ & only one eigenvector & there is an M s.t.

$M^{-1}AM = J$

Ex 4: (p. 357)

J is as close to diagonal as we can get. $d\underline{u}/dt$ cannot be further simplified by change of var.s

$$\frac{d\underline{u}}{dt} = J\underline{u} = \begin{bmatrix} 5 & 1 & \\ & 5 & 1 \\ & & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \frac{dx}{dt} = 5x + y \Rightarrow x = \left(x(0) + t\,y(0) + \tfrac{1}{2}t^2 z(0)\right)e^{5t}$$

$$\Uparrow$$

$$\frac{dy}{dt} = 5y + z \Rightarrow y = \left(y(0) + t\,z(0)\right)e^{5t}$$

$$\Uparrow$$

$$\frac{dz}{dt} = 5z \Rightarrow z = z(0)e^{5t}$$

(back substitution)

(t & t^2 enters $\because$ $\lambda = 5$ is a triple eigenvalue with only one eigenvector)

Q: If two matrices have same repeated eigenvalues & same rank $(J-\lambda I)$, are they similar to each other?

Ex:
$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad \begin{matrix} \lambda = 0, 0, 0, 0 \\ \text{rank} = 2 \end{matrix}$$

$$\dim N(A - \lambda I) = 4 - 2 = 2$$

$\Rightarrow$ two indep. eigenvectors & two "missing" eigenvectors

$$B = \begin{bmatrix} 0 & 1 & 7 & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix} \qquad \begin{matrix} \text{rank} = 2 \\ \dim N(B - \lambda I) = 2 \end{matrix}$$

& B is similar to A (Jordan form)

$$C = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \qquad \begin{matrix} \text{rank} = 2 \\ \dim N(C - \lambda I) = 2 \end{matrix}$$

Q: Is C similar to A?

No!

Jordan blocks:

$$A = \left[\begin{array}{ccc|c} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ \hline & & & 0 \end{array}\right], \quad C = \left[\begin{array}{cc|cc} 0 & 1 & & \\ & 0 & & \\ \hline & & 0 & 1 \\ & & & 0 \end{array}\right]$$

$\boxed{\text{Def}}$ A Jordan block J_i with repeated eigenvalues λ_i is

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & & & \lambda_i \end{bmatrix}$$

then the Jordan form is

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_d \end{bmatrix}$$

each block accounts for one eigenvector

(# of blocks = # of eigenvectors)

$\boxed{\text{Thm}}$ Jordan's thm

Every square matrix A is similar to a Jordan form

$\Rightarrow$ A is similar to B if they share the same Jordan form

Summary:

1. If A has n distinct eigenvalues, it is diagonalizable & $J = \Lambda$
2. If A has repeated eigenvalues & missing eigenvectors, then its J has $n-d$ ones above its diagonal