EE 203001 SES #31

Applications of PD matrices

Application I: test for minimum

2x2 Example: $A = \begin{bmatrix} 2 & 6 \\ 6 & c \end{bmatrix}$

Q: When is A a PD matrix?

Use det. test $\Rightarrow$ A is PD

when $\det A > 0$ or $2c - 36 > 0$

or $c > 18$

Case I: $c = 18$

The matrix $A = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ is on the

borderline and is a PSD matrix

A is singular & $\lambda_1 = 0$, $\lambda_2 = 20$

only has one pivot

$$\underline{x}^T A \underline{x} = [x \quad y] \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [x \quad y] \begin{bmatrix} 2x + 6y \\ 6x + 18y \end{bmatrix}$$

$$= 2x^2 + 12xy + 18y^2$$

$$= 2(x + 3y)^2 = 0 \text{ when } \begin{matrix} x = 3 \\ y = -1 \end{matrix}$$

Case II : $c < 18$ $(c = 7)$

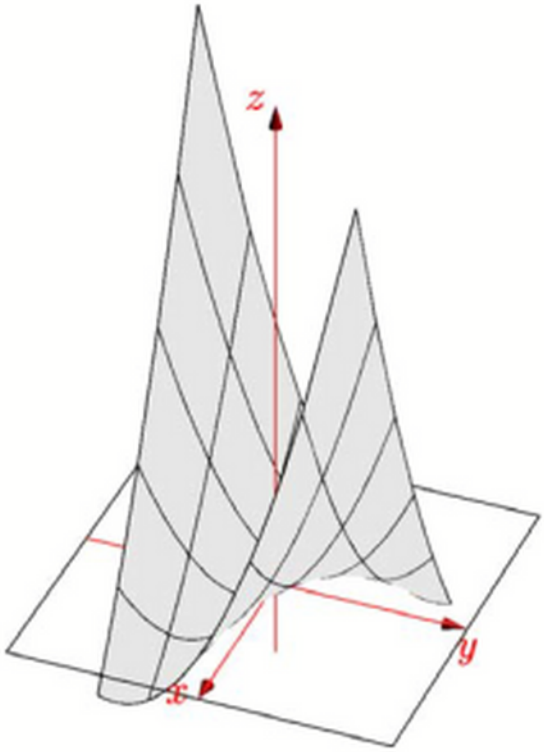

Figure 1: The graph of $f(x, y) = 2x^2 + 12xy + 7y^2$.

$A = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$ (NOT PD)

$\underline{x}^T A \underline{x} = 2x^2 + 12xy + 7y^2$

$= 2(x+3y)^2 - 11y^2$

may be negative (e.g., when $x = -3$, $y = 1$)

$(0, 0, 0)$ is a Saddle point

Case III : $c > 18$ $(c = 20)$

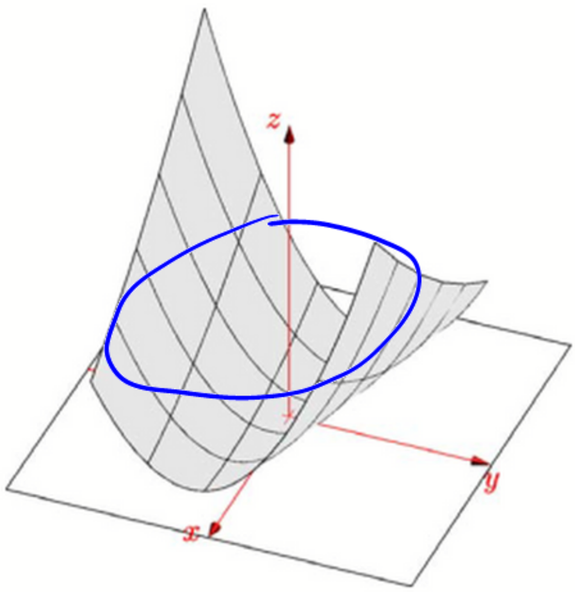

Figure 2: The graph of $f(x, y) = 2x^2 + 12xy + 20y^2$.

$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ is PD

$f(x, y) = \underline{x}^T A \underline{x}$

$= 2x^2 + 12xy + 20y^2$

$= 2(x+3y)^2 + 2y^2 > 0$

except $x = y = 0$

level curve: $f(x, y) = k$ is an ellipse

Test for minimum:

First derivative:

$$f_x = \frac{\partial f(x,y)}{\partial x} = 4x + 12y = 0 \quad \text{at } (0,0)$$

$$f_y = \frac{\partial f(x,y)}{\partial y} = 12x + 40y = 0 \quad \text{at } (0,0)$$

$\Rightarrow$ $f(x,y)$ is tangent to x-y plane at $(0,0,0)$

Q: Is this enough to show that $(0,0,0)$ is a minimum?

Not really!

This is also true for

$$f(x,y) = 2x^2 + 12y + 7y^2$$

but $(0,0,0)$ is a Saddle point, not a minimum point!

Q: What else do we need?

Second derivatives!

Recall from calculus:

If $f(0,0)$ is a minimum point

then $f_{xx} > 0$, $f_{yy} > 0$

$$f_{xx} f_{yy} - f_{xy}^2 > 0 \quad \text{at } (0,0)$$

Hessian matrix:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Note 1:

H is symmetric since $f_{xy} = f_{yx}$

Note 2:

$f(0,0)$ is a minimum point is equivalent to H is a PD matrix

(Test for upper left det:

$f_{xx} > 0$, $f_{xx} f_{yy} - f_{xy}^2 > 0$

$\Rightarrow f_{yy} > 0$)

Back to example:

$$f(x,y) = \underline{x}^T A \underline{x} = [x \; y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= ax^2 + 2bxy + cy^2$$

At $(0,0)$:

$$\left.\begin{aligned} f_{xx} &= 2a \\ f_{yy} &= 2c \\ f_{xy} &= 2b \end{aligned}\right] \Rightarrow H = \begin{bmatrix} 2a & 2b \\ 2b & 2c \end{bmatrix}$$

$$= 2 \begin{bmatrix} a & b \\ b & c \end{bmatrix} = 2A$$

So $(0,0)$ is a minimum point if

A is PD

$n \times n$

A fcn of n var.s $f(x_1, x_2, \cdots, x_n)$ has a minimum when its Hessian matrix is PD

Ex: 3×3

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Q: Is A a PD matrix?

Test 1: upperleft det

$\det(2) = 2$, $\det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 5$

$\det A = 4$ (v)

Test 2: pivots

By elimination, pivots $= 2, 3/2, 4/3$

(v)

Test 3: eigenvalues

$|A - \lambda I| = 0 \Rightarrow \lambda = 2, 2-\sqrt{2}, 2+\sqrt{2}$ (v)

Q: What is the fcn $\underline{x}^T A \underline{x}$?

$$f(\underline{x}) = \underline{x}^T A \underline{x} = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$$

(sum of squares > 0 ∵ positive pivots)

∵ A is PD ⇒ $f(\underline{x}) > 0$ except when $\underline{x} = \underline{0}$

Its graph is a sort of 4D bowl or paraboloid

Application II: Ellipsoids in $\mathbb{R}^n$

Recall: 2×2 & $f(x, y)$ is a bowl

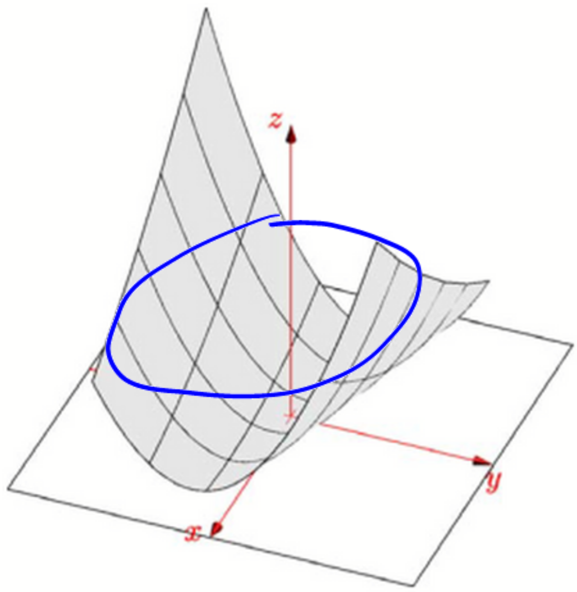

Figure 2: The graph of $f(x, y) = 2x^2 + 12xy + 20y^2$.

$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ is PD

$$f(x, y) = \underline{x}^T A \underline{x}$$
$$= 2x^2 + 12xy + 20y^2$$
$$= 2(x + 3y)^2 + 2y^2 > 0$$

except $x = y = 0$

level curve: $f(x, y) = k$ is an ellipse

In general,

$\underline{x}^T A \underline{x} = 1$ is a tilted ellipse centered at $(0, 0)$

Ex: Find the axes of the tilted ellipse $5x^2 + 8xy + 5y^2 = 1$

In the form of $\underline{x}^T A \underline{x}$:

$$[x\ y] \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ is PD

eigenvalue & eigenvector:

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 4^2$$

$$= (1-\lambda)(9-\lambda)$$

$\Rightarrow \lambda = 1, 9$

$$(A - I)\underline{x}_1 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \underline{x}_1 = \underline{0} \Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - 9I)\underline{x}_2 = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \underline{x}_2 = 0 \Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Divide by $\sqrt{2}$ to make them unit vectors

$A = Q \Lambda Q^T$

$$\Rightarrow \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\underline{x}^T A \underline{x}$ = sum of squares

$$\Rightarrow 5x^2 + 8xy + 5y^2 = 9\left(\frac{x+y}{\sqrt{2}}\right)^2 + 1\left(\frac{x-y}{\sqrt{2}}\right)^2$$

Note: compare with $A = LDL^T$

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \to \begin{bmatrix} 5 & 4 \\ 0 & \frac{9}{5} \end{bmatrix}$$ pivots: $5, \frac{9}{5}$

multiplier: $\frac{4}{5}$

$$\Rightarrow 5x^2 + 8xy + 5y^2 = 5\left(x + \frac{4}{5}y\right)^2 + \frac{9}{5}y^2$$

Note: the axes of the tilted ellipse point along the eigenvectors

(This explains why $A = Q\Lambda Q^T$ is also called "principal axis thm"

$\Rightarrow$ display both axis direction (from eigenvectors) & axis length (from eigenvalues))

Lined up:

$$\frac{x+y}{\sqrt{2}} = X, \quad \frac{x-y}{\sqrt{2}} = Y \Rightarrow 9X^2 + Y^2 = 1$$

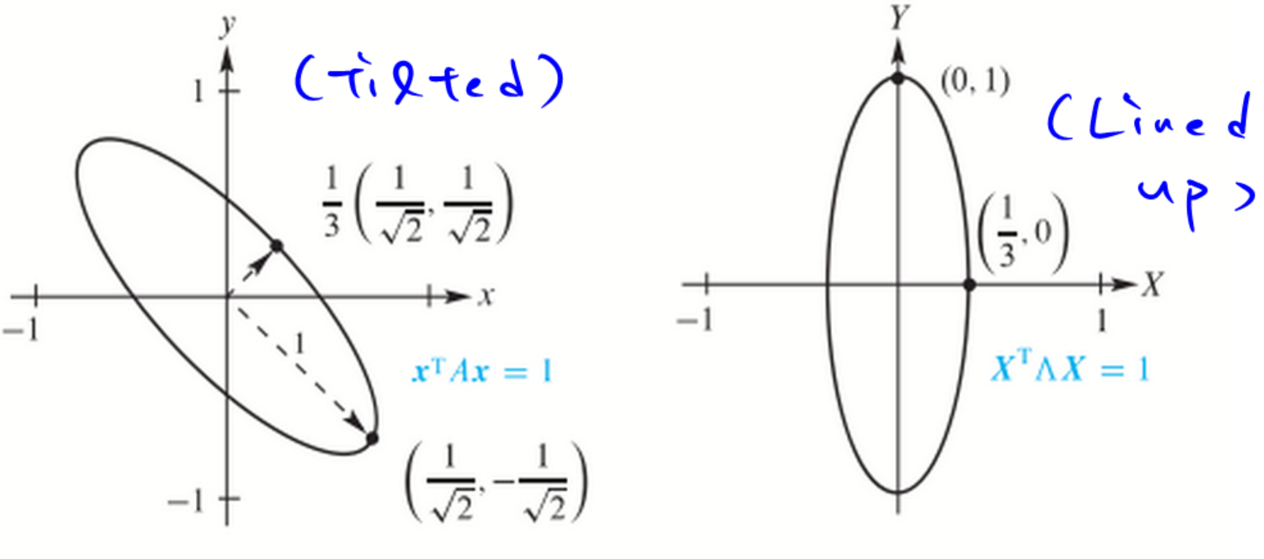

Figure 44: The tilted ellipse $5x^2 + 8xy + 5y^2 = 1$. Lined up it is $9X^2 + Y^2 = 1$.

To sum up:

Suppose $A = Q\Lambda Q^T$ is PD $\Rightarrow \lambda_i > 0$

the graph $\underline{x}^T A \underline{x} = 1$ is an ellipse:

$$[x\ y]\, Q\Lambda Q^T \begin{bmatrix} x \\ y \end{bmatrix} = [X\ Y]\, \Lambda \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$= \lambda_1 X^2 + \lambda_2 Y^2 = 1$$

(The axes point along eigenvectors, the half-lengths are $1/\sqrt{\lambda_1}$ & $1/\sqrt{\lambda_2}$)

Note:

$A = I$ gives the circle $x^2 + y^2 = 1$

Note:

If one eigenvalue < 0, we don't have an ellipse:

Sum of squares becomes a diff. of squares: $9X^2 - Y^2 = 1$, we get a hyperbola

If all $\lambda < 0$, e.g. $A = -I$

$-x^2 - y^2 = 1$ has no points at all

(Can be generalized to $n \times n$ $\Rightarrow$ Ellipsoids in R^n)