EE203001 SES #30

Positive definite matrices

Studying positive definite matrices brings the whole course together: pivots / determinants / eigenvalues / stability

(PD)

$\boxed{\text{Def}}$ A matrix is positive definite if: 1. the matrix is symmetric

2. All $\lambda > 0$

Note: if $\lambda \geq 0$, we have a positive semidefinite matrix

(PSD)

Issue: Computing eigenvalues is a lot of work!

Q: Can we have a quick test?

Yes!

Start with 2x2

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ When does A have $\lambda_1 > 0,\ \lambda_2 > 0$?

Note: λ's are real $\because$ A is symmetric

$\boxed{\text{Fact}}$ The eigenvalues of A are positive

iff $a > 0$ & $ac - b^2 > 0$

(upper left determinant)

proof:

"$\Rightarrow$" If $\lambda_1 > 0$, $\lambda_2 > 0$, then

$$ac - b^2 = \det A = \lambda_1 \lambda_2 > 0$$

$$a + c = \text{trace } A = \lambda_1 + \lambda_2 > 0$$

$\Rightarrow$ a, c both positive

(If not, $ac - b^2 > 0$ fail)

"$\Leftarrow$" If $a > 0$, $ac - b^2 > 0$, then

$$c > b^2/a > 0$$

so $\lambda_1 \lambda_2 = \det A = ac - b^2 > 0$

$\lambda_1 + \lambda_2 = \text{trace } A = a + c > 0$

$\Rightarrow$ $\lambda_1 > 0$, $\lambda_2 > 0$

Ex:

$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $a > 0$, but $ac - b^2 = 1 - 4 < 0$

(x)

$A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$, $a > 0$ & $ac - b^2 = 6 - 4 > 0$

(v)

$A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix}$, $ac - b^2 = 6 - 4 > 0$, but $a < 0$

(x)

$\boxed{\text{Fact}}$ The eigenvalues of $A = A^T$ are positive iff the pivots are positive, i.e.,

$$a > 0 \ \& \ \frac{ac - b^2}{a} > 0$$

proof: recall for symmetric matrices

\# of positive eigenvalues = \# of positive pivots

Chk the pivots:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \to \begin{bmatrix} a & b \\ 0 & c - \frac{b}{a} b \end{bmatrix}$$ pivots: a, $c - \frac{b^2}{a} = \frac{ac - b^2}{a}$

(This is a lot faster than computing eigenvalues!)

Back to example:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 2 \\ 2 & -6 \end{bmatrix}$$

pivots: $1 \ \& \ -3$ (indefinite) $\qquad 1 \ \& \ 2$ (positive definite) $\qquad -1 \ \& \ -2$ (negative definite)

Energy-based definition

If λ's > 0, from $A\underline{x} = \lambda \underline{x}$

$\Rightarrow \underline{x}^T A \underline{x} = \lambda \underline{x}^T \underline{x} = \lambda \|\underline{x}\|^2 > 0$

(true for any eigenvector)

New idea: Not just for eigenvectors but $\forall$ nonzero vectors $\underline{x}$

$\underline{x}^T A \underline{x} > 0$ (energy of the system)

$\boxed{\text{Def}}$ (The common definition of PD)

A is PD if $\underline{x}^T A \underline{x} > 0$ for all nonzero $\underline{x}$:

$$\underline{x}^T A \underline{x} = [x \ \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \underbrace{ax^2 + 2bxy + cy^2}_{} > 0$$

(from off-diagonal b & b) (from diagonal a, c)

$\boxed{\text{Fact}}$ If A, B are PD, so is A+B

proof: $\underline{x}^T (A+B) \underline{x} = \underline{x}^T A \underline{x} + \underline{x}^T B \underline{x} > 0$

$\Rightarrow$ A+B is PD

(pivots & eigenvalues are not easy to follow when matrices are added, but the energies just add!)

$\boxed{\text{Fact}}$ If the col.s of R are indep.

then $A = R^T R$ is PD

(R can be rectangular, but $A = R^T R$ is square & symmetric)

proof: $\underline{x}^T A \underline{x} = \underline{x}^T R^T R \underline{x} = (R\underline{x})^T (R\underline{x})$

$= \|R\underline{x}\|^2$

$R\underline{x} \neq 0$ when $\underline{x} \neq \underline{0}$ if col.s of R are indep. $\Rightarrow \underline{x}^T A \underline{x} > 0$

$\Rightarrow A$ is PD

$\boxed{\text{Statements}}$ (Five equivalent statements of PD)

When a symmetric matrix is PD following statements are equivalent

1. All n pivots > 0
2. All upper left determinant > 0
3. All n eigenvalues > 0
4. $\underline{x}^T A \underline{x} > 0$ except at $\underline{x} = 0$ (energy-based def.)
5. $A = R^T R$ & R has indep. col.s

Q: How to link 1–3 with 4–5?

Show by an example (almost a proof)

Ex: Test A & B for PD

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

For A:

1. pivots: $2, \frac{3}{2}, \frac{4}{3}$ (multiplier: $-\frac{1}{2}, -\frac{2}{3}$)
2. upper left det: $2, 3, 4$
3. eigenvalues: $2-\sqrt{2}, 2, 2+\sqrt{2}$
4. $\underline{x}^T A \underline{x} = [x_1\ x_2\ x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$= 2(x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + x_3^2)$$

(complete the squares) $= 2\left(x_1 - \frac{1}{2}x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3}x_3\right)^2 + \frac{4}{3}x_3^2$

pivots: $2, \frac{3}{2}, \frac{4}{3}$

multipliers: $-\frac{1}{2}, -\frac{2}{3}$

> 0, If all pivots > 0

Q: Is this a coincidence?

No, we will see why later

5. $A = R^T R$

choice one:

$$A = LDL^T$$ (symmetric version of LU)

$\Rightarrow A = LDL^T = (L\sqrt{D})(L\sqrt{D})^T = R^T R$

(cholesky factor)

$\Rightarrow$ A is PD since L^T has indep. col.s

Specifically,

$$A = LDL^T = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix}$$

$$\Rightarrow \underline{x}^T A \underline{x} = \underline{x}^T L \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \underline{x}^T L \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \overbrace{\begin{bmatrix} x_1 - \frac{1}{2} x_2 \\ x_2 - \frac{2}{3} x_3 \\ x_3 \end{bmatrix}}^{L^T \underline{x}}$$

$$= 2\left(x_1 - \frac{1}{2} x_2\right)^2 + \frac{3}{2}\left(x_2 - \frac{2}{3} x_3\right)^2 + \frac{4}{3} x_3^2$$

choice two:

$$A = R^T R \quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

(first diff. matrix)

For B: Determinant test is easiest

$\Rightarrow$ only need to chk

$\det B = 4 + 2b - 2b^2 = (1+b)(4-2b)$

At $b=-1$ & $b=2$, $\det B = 0$

$-1 < b < 2 \Rightarrow \det B > 0 \Rightarrow B$ is PD

Positive semidefinite matrices (PSD)

At the edge of PD, $\underline{x}^T A \underline{x} \geq 0$

or smallest eigenvalues $= 0$ or $\det = 0$

Ex: $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ $\det A = 0$

$A\underline{x} = \underline{0} \Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ eigenvector

$\underline{x}^T A \underline{x} = 0$ for this eigenvector

$\underline{x}^T A \underline{x} > 0$ for all other directions

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\text{dependent col.s}} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(Cyclic A from cyclic R)