Complex matrices

Matrices with all real entries can still have complex eigenvalues

$\Rightarrow$ We cannot avoid dealing with complex numbers !

Complex vectors

Length:

Given a vector $\underline{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$

with complex entries

Q: How do we find it length ?

Our old definition:

$$\underline{z}^T \underline{z} = [z_1 \cdots z_n] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$ (No good !) (Not always positive !)

Ex: $\underline{z} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$\underline{z}^T \underline{z} = [1 \quad i] \begin{bmatrix} 1 \\ i \end{bmatrix} = 0 \quad ?$$

Correct definition:

$$\|\underline{z}\|^2 = \bar{\underline{z}}^T \underline{z} = [\bar{z}_1 \cdots \bar{z}_n] \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

$$= |z_1|^2 + \cdots + |z_n|^2 \geq 0$$

Ex:

$$\left(\text{length} \begin{bmatrix} 1 \\ i \end{bmatrix}\right)^2 = [1 \ \ -i] \begin{bmatrix} 1 \\ i \end{bmatrix} = 2 \ (\checkmark)$$

Simplify notition:

$$\|\underline{z}\|^2 = \underline{z}^H \underline{z} \quad \text{where } \underline{z}^H = \bar{\underline{z}}^T$$

Same for matrices:

$$\text{If } A = \begin{bmatrix} 1 & i \\ 0 & 1+i \end{bmatrix}, \ A^H = \begin{bmatrix} 1 & 0 \\ -i & 1-i \end{bmatrix}$$

<u>Inner product</u>

$$\underline{y}^H \underline{x} = \bar{\underline{y}}^T \underline{x} = \bar{y}_1 x_1 + \cdots + \bar{y}_n x_n$$

Note:

$$\underline{y}^H \underline{x} \neq \underline{x}^H \underline{y} = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$$

$$= \text{complex conjugate of } \underline{y}^H \underline{x}$$

(order is important!)

Ex: $\underline{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \ \underline{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$$\underline{u}^H \underline{v} = [1 \ \ -i] \begin{bmatrix} i \\ 1 \end{bmatrix} = 0 \ \text{(orthogonal)}$$

Note: $(A\underline{u})^H \underline{v} = \underline{u}^H (A^H \underline{v})$

Reason: $(A\underline{u})^H = \overline{A\underline{u}}^T = \overline{\underline{u}}^T \overline{A}^T = \underline{u}^H A^H$

(Inner product of $A\underline{u}$ with $\underline{v}$ equals Inner product of $\underline{u}$ with $A^H\underline{v}$)

Note: $(AB)^H = B^H A^H$

Hermitian matrices

Recall: For symmetric matrix $A = A^T$

$\Rightarrow$ real eigenvalues

$\Rightarrow$ there is a full set of orthogonal eigenvectors

$\Rightarrow$ Diagonalising matrix $S = Q$ (orthogonal)

$\Rightarrow$ $A = Q \Lambda Q^{-1}$ or $A = Q \Lambda Q^T$

(All this follows from $a_{ij} = a_{ji}$ when A is real)

Now for complex matrices

We have Hermitian matrix $A = A^H$

where $a_{ij} = \overline{a_{ji}}$

Note: Every symmetric matrix is Hermitian

($a_{ij} = a_{ji} = \overline{a_{ji}}$ for real a_{ji})

Ex: Hermitian matrix

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} = A^H$$

$\boxed{\text{Fact}}$ If $A = A^H$ and $\underline{z}$ is any vector then $\underline{z}^H A \underline{z}$ is real

proof: $\underline{z}^H A \underline{z}$ is 1×1 number

$$\Rightarrow (\underline{z}^H A \underline{z})^H = \underline{z}^H A^H (\underline{z}^H)^H = \underline{z}^H A \underline{z}$$

the number is real since it is equal to its conjugate

Back to example:

$$[\bar{z}_1 \ \bar{z}_2] \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \underbrace{2\bar{z}_1 z_1 + 5\bar{z}_2 z_2}_{\text{(diagonal)}} + (3-3i)\bar{z}_1 z_2$$
$$+ \underbrace{(3+3i) z_1 \bar{z}_2}_{\text{(off-diagonal)}}$$

($2|z_1|^2$ & $5|z_2|^2$ are both real

the off-diagonal terms are conjugate of each other $\Rightarrow$ sum is real)

$\boxed{\text{Fact}}$ Every eigenvalue of a Hermitian matrix is real

proof: Suppose $A\underline{z} = \lambda \underline{z}$

$$\Rightarrow \underbrace{\underline{z}^H A \underline{z}}_{\text{real}} = \lambda \underline{z}^H \underline{z} = \lambda |\underbrace{\underline{z}}_{\text{real}}|^2$$

So λ must be real !

Back to example:

$$\begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 - |3+3i|^2$$

$$= \lambda^2 - 7\lambda + 10 - 18$$

$$= (\lambda - 8)(\lambda + 1)$$

$$\Rightarrow \lambda = 8 \ \& \ -1$$

$\boxed{\text{Fact}}$ The eigenvectors of a Hermitian matrix are orthogonal (when they correspond to diff. eigenvalues)

If $A\underline{z} = \lambda \underline{z}$ & $A\underline{y} = \beta \underline{y}$ & $\lambda \neq \beta$

then $\underline{y}^H \underline{z} = 0$

proof:

$$A\underline{z} = \lambda \underline{z} \Rightarrow \underline{y}^H A \underline{z} = \lambda \underline{y}^H \underline{z}$$

$$\underline{y}^H A^H = \beta \underline{y}^H \Rightarrow \underline{y}^H A^H \underline{z} = \beta \underline{y}^H \underline{z}$$

$$\Rightarrow (\lambda - \beta) \underline{y}^H \underline{z} = 0 \Rightarrow \underline{y}^H \underline{z} = 0 \text{ if } \lambda \neq \beta$$

Back to example:

$$(A - 8I)\underline{z} = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{z} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$(A + I)\underline{y} = \begin{bmatrix} 3 & 3-3i \\ 3+3i & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{y} = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$$

$$\Rightarrow \underline{y}^H \underline{z} = [\,1+i \quad -1\,] \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = 0$$

Note: Eigenvectors have length $\sqrt{3}$

After dividing by $\sqrt{3}$, they are orthonormal

$\Rightarrow$ They go into eigenvector matrix S

that diagonalize A

(When A is real & symmetric,

S is Q - orthogonal

When A is complex & Hermitian

eigenvectors are complex & orthonormal

$\Rightarrow$ S is like Q but complex)

(Complex & orthogonal $\Rightarrow$ unitary)

Unitary matrices

A unitary matrix U is a complex square matrix that has orthonormal col.s

(U is a complex equivalent of Q)

Ex: Eigenvector matrix of A

$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

Recall: For orthonormal matrix Q (real)

$$Q^T Q = I$$

Q: What does it mean for complex vectors $\underline{q}_1, \dots, \underline{q}_n$ to be orthonormal?

Use new definition of inner product

$$\Rightarrow \quad \underline{q}_j^H \underline{q}_k = \begin{cases} 0 & , j \neq k \\ 1 & , j = k \end{cases}$$

$$Q = [\, \underline{q}_1 \cdots \underline{q}_n \,] \Rightarrow Q^H Q = I$$

$\boxed{\text{Fact}}$ Every matrix U with orthonormal col.s has $U^H U = I$

If U is square, then $U^H = U^{-1}$

$\boxed{\text{Fact}}$ If U is unitary, then $\|U\underline{z}\| = \|\underline{z}\|$

$\Rightarrow U\underline{z} = \lambda \underline{z}$ leads to $|\lambda| = 1$

proof: $\|U\underline{z}\|^2 = \underline{z}^H U^H U \underline{z} = \underline{z}^H \underline{z} = \|\underline{z}\|^2$

Back to example:

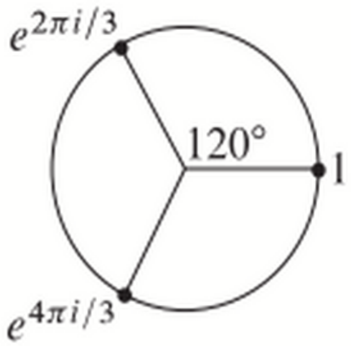
$$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$ both Hermitian & unitary

$\Rightarrow$ real eigenvalues & $|\lambda| = 1$

$\Rightarrow \lambda = 1$ or -1

since trace $= 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -1$

Ex: 3×3 Fourier matrix



Fourier matrix $$F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

Figure 61: The cube roots of 1 go into the Fourier matrix $F = F_3$.

Q: Is it Hermitian?

$$F^H = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{-4\pi i/3} \\ 1 & e^{-4\pi i/3} & e^{-2\pi i/3} \end{bmatrix} \neq F$$

Q: Is it unitary?

The squared length of each col.

$= \frac{1}{3}(1+1+1) = 1$ (unit vectors)

$$(\text{col } 1)^H (\text{col. } 2) = \frac{1}{3}\left(1 + e^{2\pi i/3} + e^{4\pi i/3}\right)$$
$$= 0$$

$$(\text{col } 2)^H (\text{col. } 3) = \frac{1}{3}\left(1 \cdot 1 + e^{-2\pi i/3} e^{4\pi i/3} + e^{-4\pi i/3} e^{2\pi i/3}\right) = \frac{1}{3}\left(1 + e^{2\pi i/3} + e^{-2\pi i/3}\right)$$
$$= 0$$

$\Rightarrow F$ is unitary!

(Read real v.s. complex, p.506)