EE 203001 SBS #27

Symmetric matrices

Recall: A is symmetric if $A^T = A$

If a matrix has special properties (e.g., Markov matrices), its eigenvalues & eigenvectors are likely to have special properties

Q: What is speial about $A\underline{x} = \lambda \underline{x}$ if A is symmetric?

$\boxed{\text{Fact}}$ For a symmetric matrix with real entries, we have

1. All eigenvalues are real
2. Eigenvectors can be chosen to be orthonormal

Note: Every symmetric matrix can be diagonalized (will prove this later when repeated eigenvalues)

Note: Its eigenvector matrix S becomes an orthogonal matrix Q where $Q^{-1} = Q^T$

This leads to the Spetral Thm

$\boxed{\text{Spetral Thm}}$ Every symmetric matrix has the factorization $A = Q \Lambda Q^T$ with real eigenvalues in Λ and orthonormal eigenvectors in $S = Q$

Note: Easy to see $Q \Lambda Q^T$ is symmetric

Any $A = Q \Lambda Q^T$ is symmetric

Note: This is "spectral thm" in math & "principal axis thm" in mechanics and physics

Reason: Approach in 3 steps

Step 1: By an example, showing real λ's in Λ & orthonormal x in Q

Step 2: By a proof when no repeated eigenvalues

Step 3: By a proof that allows repeated eigenvalues

Ex 1 (p. 331) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|A - \lambda I| = \lambda^2 - 5\lambda \Rightarrow \lambda = 0$ or 5

(Can see this directly: A is singular

$\Rightarrow \lambda_1 = 0$ is an eigenvalue. $\mathrm{tr} A = 1 + 4 = 5$

$\Rightarrow \lambda_1 + \lambda_2 = 5 \Rightarrow \lambda_2 = 5$)

Eigenvectors:

$A \underline{x}_1 = \underline{0} \Rightarrow \underline{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$(A - 5I)\underline{x}_2 = \underline{0} \Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q: Why $\underline{x}_1$ & $\underline{x}_2$ are orthogonal?

$\underline{x}_1$ in $N(A)$, $\underline{x}_2$ in $C(A)$

($A\underline{x}_2 = 5\underline{x}_2 \Rightarrow \underline{x}_2$ is a comb. of

col.s of $A \Rightarrow \underline{x}_2 \in C(A)$)

Q: $N(A) \perp C(A^T)$ not $C(A)$?

But A is symmetric $\Rightarrow A^T = A$

$\Rightarrow C(A^T) = C(A)$ (row space = col. space)

Normalize $\underline{x}_1$ & $\underline{x}_2$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = Q \Lambda Q^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

$\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (v)

$\boxed{\text{Fact}}$ All eigenvalues of real symmetric matrix are real

proof: $A\underline{x} = \lambda\underline{x} \Rightarrow \bar{A}\bar{\underline{x}} = \bar{\lambda}\bar{\underline{x}}$

$(\lambda = a + ib,\ \bar{\lambda} = a - ib)$

$\Rightarrow A\bar{\underline{x}} = \bar{\lambda}\bar{\underline{x}}$ (A is real)

(complex eigenvalues of real A always comes in conjugate pairs)

Take transpose

$\Rightarrow \bar{\underline{x}}^T A^T = \bar{\underline{x}}^T \bar{\lambda}$

$\Rightarrow \bar{\underline{x}}^T A = \bar{\underline{x}}^T \bar{\lambda}$ $(A = A^T)$

Multiply by $\underline{x}$ on the right

$\Rightarrow \bar{\underline{x}}^T A \underline{x} = \bar{\underline{x}}^T \bar{\lambda} \underline{x}$ — ①

On the other hand, $A\underline{x} = \lambda\underline{x}$

Multiply by $\bar{\underline{x}}^T$ on the left

$\Rightarrow \bar{\underline{x}}^T A \underline{x} = \bar{\underline{x}}^T \lambda \underline{x}$ — ②

Comparing ① & ②

$\Rightarrow \lambda \bar{\underline{x}}^T \underline{x} = \bar{\lambda} \bar{\underline{x}}^T \underline{x}$

$\Rightarrow \lambda = \bar{\lambda}$ if $\bar{\underline{x}}^T \underline{x} \neq 0$

$$\left(\bar{\underline{x}}^T \underline{x} = [\bar{x}_1 \ \cdots \ \bar{x}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = |x_1|^2 + \cdots + |x_n|^2 \right.$$

$\Rightarrow \bar{\underline{x}}^T \underline{x} \neq 0$ if $\underline{x} \neq \underline{0}$)

Note: eigenvectors come from solving $(A - \lambda I)\underline{x} = \underline{0}$ since λ all real

$\Rightarrow$ eigenvectors all real

$\boxed{\text{Fact}}$ Eigenvectors of a real symmetric matrix (corr. to diff. eigenvalues) are always perpendicular

proof:

Let $A\underline{x} = \lambda_1 \underline{x}$, $A\underline{y} = \lambda_2 \underline{y}$

$\Rightarrow (A\underline{x})^T \underline{y} = \lambda_1 \underline{x}^T \underline{y}$, $\underline{x}^T A \underline{y} = \lambda_2 \underline{x}^T \underline{y}$

$$\Rightarrow \left.\begin{array}{l} \underline{x}^T A^T \underline{y} = \lambda_1 \underline{x}^T \underline{y} \\ \quad \parallel \\ \underline{x}^T A \underline{y} = \lambda_2 \underline{x}^T \underline{y} \end{array}\right\} \Rightarrow \underline{x}^T \underline{y} = 0$$

$(\lambda_1 \neq \lambda_2)$

$\Rightarrow$ eigenvector for λ_1 $\perp$ eigenvector for λ_2

(True for any pair)

Note: If A has complex entries, A has real eigenvalues & perpendicular eigenvectors iff $A = \bar{A}^T$

(Proof of this follows same pattern)

Projection onto eigenvectors

If $A = A^T$, we have

$$A = Q\Lambda Q^T$$

$$= \begin{bmatrix} \underline{q}_1 & \cdots & \underline{q}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \underline{q}_1^T \\ \vdots \\ \underline{q}_n^T \end{bmatrix}$$

$$= \lambda_1 \underline{q}_1 \underline{q}_1^T + \cdots + \lambda_n \underline{q}_n \underline{q}_n^T$$

$$= \lambda_1 P_1 + \cdots + \lambda_n P_n$$

(projection onto eigenvectors)

(Every symmetric matrix is a comb. of perpendicular projection matrices)

Eigenvalues v.s. Pivots

For eigenvalues, we solve $\det(A - \lambda I) = 0$

For pivots, we use Elimination

(very different !)

Only connection so far:

Product of pivots = determinant

= Products of eigenvalues

For symmetric matrices,

\# of positive eigenvalues = \# of positive pivots

Special case: A has all $\lambda_i > 0$ iff all pivots are positive

Note: For large matrix, it is impractical to compute $|A - \lambda I| = 0$

But NOT hard to compute pivots by elimination

$\Rightarrow$ can use signs of pivots to determine signs of λ

e.g., eigenvalues of $A - bI$ are b less than eigenvalues of A

chk pivots > 0 or < 0

$\Rightarrow \lambda - b > 0$ or < 0

$\Rightarrow \lambda > b$ or $\lambda < b$

(we can chk whether $\lambda > b$ or $\lambda < b$ for any b!)

Now, we try to show that even for repeated eigenvalues, $A = A^T$ has perpendicular eigenvectors

$\boxed{\text{Fact}}$ Every square matrix factors into

$A = QTQ^{-1}$ where T: upper triangular

$\bar{Q}^T = Q^{-1}$

If A has real eigenvalues, then Q & T can be chosen to be real: $Q^T Q = I$

$\boxed{\text{Fact}}$ Eigenvectors of a real symmetric matrix (even with repeated eigenvalues) are always perpendicular

Proof: For symmetric matrix A, eigenvalues are all real $\Rightarrow A = QTQ^T$, $Q^T Q = I$

$\Rightarrow Q^T A Q = Q^T Q T Q^T Q = T$

Since $A = A^T \Rightarrow T = T^T$ but T is upper-triangular $\Rightarrow T = \Lambda$ is diagonal

$\Rightarrow A = Q \Lambda Q^T$ for orthogonal Q

$\Rightarrow A$ has orthonormal eigenvectors