# 2017 Fall EE203001 Linear Algebra - Midterm 1 solution

1. (10%)

(a) $\begin{bmatrix} 1 & 2 & 3 & 31 \\ 2 & 3 & 5 & 69 \\ 3 & 5 & a & b \end{bmatrix}$

(b) a=8

(c) a≠8

(d) a=8 , b=100

(e) a=8 , b≠100

2. (15%)

(a) $\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & -6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -5 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & -1 & -4 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & -1 & -5 \end{bmatrix}$$

(b) $A = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ \hline 0 & 0 & -5 & -6 \\ 0 & 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{c|c} B_{2\times 2} & C_{2\times 2} \\ \hline O_{2\times 2} & D_{2\times 2} \end{array}\right]$

$$\begin{bmatrix} B & C & I & O \\ O & D & O & I \end{bmatrix} \rightarrow \begin{bmatrix} I & B^{-1}C & B^{-1} & O \\ O & I & O & D^{-1} \end{bmatrix} \rightarrow \begin{bmatrix} I & O & B^{-1} & -B^{-1}CD^{-1} \\ O & I & O & D^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ O & D^{-1} \end{bmatrix}$$

(c) $B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, D^{-1} = \begin{bmatrix} 1 & 6 \\ -1 & -5 \end{bmatrix}$

$$-B^{-1}CD^{-1} = -\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}CD^{-1} \\ O & D^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & -1 & -5 \end{bmatrix}$$

3. (20%)

(a) $P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$PA = E_{21}^{-1}E_{32}^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} = LU$$

(b) $PA\boldsymbol{x} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} \Rightarrow LU\boldsymbol{x} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\boldsymbol{c} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$

$\Rightarrow \boldsymbol{c} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

4. (20%)

(a) Reduce A to either $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ or its RREF $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

$\Rightarrow$ a basis for

**Row(A)** $= \{(1,2,0,3),(0,1,3,0),(0,0,1,0)\}$ or

$\{(1,0,0,3),(0,1,0,0),(0,0,1,0)\}$ or $\{(1,2,0,3),(2,5,3,6),(1,3,4,3)\}$

$\Rightarrow$ a basis for

**Col(A)** $= \{(1,2,1)^T,(2,5,3)^T,(0,3,4)^T\}$ .

(b) Yes, since rank(A)=3=m.

(c) No,since $rank(A) = 3 \neq n = 4$

(d) $rank(B) = n - dim(N(B)) = 4 - 3 = 1 \qquad dim(N(B^T)) = 5 - rank(B) = 4$

5. (20%)

(a) $[A|b] = \begin{bmatrix} 1 & 6 & 2 & 10 & 8 \\ 2 & 13 & 4 & 22 & 17 \\ -2 & -11 & -4 & -18 & -15 \\ 5 & 31 & 10 & 52 & 41 \\ 7 & 33 & 10 & 56 & 43 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$

pivot variables : $x_1$, $x_2$, $x_3$
free variables : $x_4$

(b) Find $x_p$ :
set $x_4 = 0$, apply backward substitution to $R\vec{x} = \vec{b}$

$$\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 1 \end{cases}$$

$$\rightarrow \vec{x}_p = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

(c) Find special solution : set $x_4 = c$, apply backward substitution to $R\vec{x} = \vec{0}$

$$\begin{cases} x_1 = 0 \\ x_2 + 2x_4 = 0 \\ x_3 - x_4 = 0 \end{cases}$$

$$\rightarrow \vec{x}_h = c\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

(d) $\vec{x}_{complete} = \vec{x}_p + \vec{x}_h = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

(e) $dim(row(A)) = dim(col(A)) = 3$

$dim(N(A)) = 4 - rank(A) = 4 - 3 = 1$

$dim(N(A^T)) = 5 - rank(A^T) = 5 - rank(A) = 5 - 3 = 2$

(f) basis of $col(A)$ : any three independent vectors in $span\{$*first three columns of* $A\}$

basis of $row(A)$ : any three independent vectors in $span\{$*row 1,2,5 of* $A\}$

basis of $N(A)$ : any vector in $span\{\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}\}$

basis of $N(A^T)$ : For $N(A^T)$, we can find $E$ and $R$ such $EA = R$

$$[I|A] = \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 6 & 2 & 10 \\ 0 & 1 & 0 & 0 & 0 & 2 & 13 & 4 & 22 \\ 0 & 0 & 1 & 0 & 0 & -2 & -11 & -4 & -18 \\ 0 & 0 & 0 & 1 & 0 & 5 & 31 & 10 & 52 \\ 0 & 0 & 0 & 0 & 1 & 7 & 33 & 10 & 56 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 6 & 2 & 10 \\ -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 4 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -25 & 9 & 0 & 0 & 1 & 0 & 0 & -4 & 4 \end{array}\right] = [E|R]$$

$$N(A^T) = span\{\begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\}$$

6. (15%)
(1)

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(2)

$$A^TCAx = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & -1 & -1 \\ -2 & 6 & -2 & -2 \\ -1 & -2 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Ground node four, then the equation is reduced to:

$$\begin{bmatrix} 4 & -2 & -1 \\ -2 & 6 & -2 \\ -1 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 4 & -2 & -1 & 1 \\ -2 & 6 & -2 & 0 \\ -1 & -2 & 4 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -0.5 & -0.25 & 0.25 \\ 0 & 1 & -0.5 & 0.1 \\ 0 & 0 & 1 & 0.2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0 & 0.2 \\ 0 & 0 & 1 & 0.2 \end{bmatrix}$$

$$\rightarrow \mathbf{x} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.2 \\ 0 \end{bmatrix}, \mathbf{y} = -CA\mathbf{x} = \begin{bmatrix} -0.4 \\ -0.2 \\ -0 \\ -0.4 \\ -0.4 \\ -0.2 \end{bmatrix}$$