2017 Fall EE203001 Linear Algebra - Midterm 2 solution

1. (a) $\begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 1 & -2 & -1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$

   Let $x_3 = c_3$, $x_4 = c_4$

   $S = c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is the basis for the subspaces S.

   $S^{\perp} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ is the basis for the subspaces $S^{\perp}$.

   (b) $\boldsymbol{b}_1 = \frac{<(4,6,-2,-2),(1,0,1,0)>}{\|(1,0,1,0)\|}(1,0,1,0) + \frac{<(4,6,-2,-2),(0,1,0,1)>}{\|(0,1,0,1)\|}(0,1,0,1)$

   $= 1 \times (0,1,-1,-1) + 2 \times (0,1,0,1) = (1,2,1,2)$

   $\boldsymbol{b}_2 = \boldsymbol{b} - \boldsymbol{b}_1 = (4,6,-2,-2) - (1,2,1,2) = (3,4,-3,-4)$

   (c) $f(x) = -x,\ -\pi \leq x \leq \pi$

   $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0$

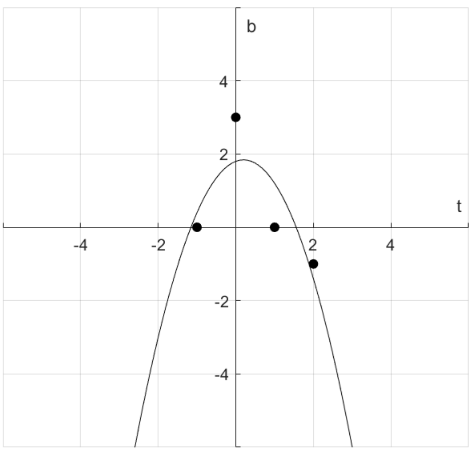
   $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(kx)dx = 0$

   $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(kx)dx = \frac{2}{k}(-1)^k$

   (d) $\|f\|^2 = \int_{-\pi}^{\pi} x^2 dx = \frac{2}{3}\pi^3$

   The length of $f(x)$ is $\sqrt{\frac{2}{3}\pi^3}$

2. (a)



$$\begin{array}{ccccccc} C & - & D & + & E & = & 0 \\ C & + & 2D & + & 4E & = & -1 \\ C & & & & & = & 3 \\ C & + & D & + & E & = & 0 \end{array}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ \hat{\mathbf{x}} = \begin{bmatrix} C \\ D \\ E \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

(b)

(1) Let $\mathbf{a}_n, \mathbf{q}_n, \mathbf{r}_n$ denote the n_{th} column of A, Q and R.

First we choose $\mathbf{q}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. $\mathbf{a}_1^T\mathbf{q}_1 = 2 \Rightarrow \mathbf{r}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

(2) $\mathbf{a}_2^T\mathbf{q}_1 = 1, \mathbf{a}_2 - \mathbf{q}_1 = \frac{1}{2}\begin{bmatrix} -3 \\ 3 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \mathbf{q}_2 = \frac{1}{2\sqrt{5}}\begin{bmatrix} -3 \\ 3 \\ -1 \\ 1 \end{bmatrix}$, and with $\mathbf{a}_2^T\mathbf{q}_2 = \sqrt{5} \Rightarrow \mathbf{r}_2 = \begin{bmatrix} 1 \\ \sqrt{5} \\ 0 \end{bmatrix}$.

(3) $\mathbf{a}_3^T\mathbf{q}_1 = 3, \mathbf{a}_3^T\mathbf{q}_2 = \sqrt{5}, \mathbf{a}_3 - 3\mathbf{q}_1 - \sqrt{5}\mathbf{q}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \Rightarrow \mathbf{q}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, and with $\mathbf{a}_3^T\mathbf{q}_3 = 2 \Rightarrow \mathbf{r}_3 = \begin{bmatrix} 3 \\ \sqrt{5} \\ 2 \end{bmatrix}$. So $A = QR = \left(\frac{1}{2}\begin{bmatrix} 1 & \frac{-3}{\sqrt{5}} & 1 \\ 1 & \frac{3}{\sqrt{5}} & 1 \\ 1 & \frac{-1}{\sqrt{5}} & -1 \\ 1 & \frac{1}{\sqrt{5}} & -1 \end{bmatrix}\right)\begin{bmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}$.

(c)

Starting from $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \Rightarrow A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$. Now we know that $A = QR \Rightarrow A^TA = R^TQ^TQR = R^TR \Rightarrow A^TA\hat{\mathbf{x}} = R^TR\hat{\mathbf{x}} = A^T\mathbf{b} = R^TQ^T\mathbf{b} \Rightarrow R\hat{\mathbf{x}} = Q^T\mathbf{b}$.

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}\hat{\mathbf{x}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-3}{\sqrt{5}} \\ -2 \end{bmatrix} \Rightarrow \hat{\mathbf{x}} = \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1.8 \\ 0.4 \\ -1 \end{bmatrix}.$$

The distances to the parabola $b = 1.8 + 0.4t - t^2$ for each point is:

$$\begin{aligned} \mathbf{e} &= \mathbf{b} - \mathbf{p} \\ &= \mathbf{b} - A\hat{\mathbf{x}} \\ &= \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.4 \\ -1.4 \\ 1.8 \\ 1.2 \end{bmatrix} \\ &= \begin{bmatrix} -0.4 \\ 0.4 \\ 1.2 \\ -1.2 \end{bmatrix} \end{aligned}$$

The total vertical distance is the sum of absolute values of all elements of $\mathbf{e}$, which is 3.2.

(d)
Since the new column vector of Q must be orthogonal to all the former column vectors , $\mathbf{q}$ should lie on $\mathbf{a} - QQ^T\mathbf{a}$, which is $\mathbf{a}$ subtracts it's projection onto the column space of Q. So $\mathbf{q} = \frac{\mathbf{a}-QQ^T\mathbf{a}}{\|\mathbf{a}-QQ^T\mathbf{a}\|}$.

3. (a) $det(\boldsymbol{C}) = 25$

(b) $det(\boldsymbol{C}) = [det(\boldsymbol{A})]^{3-1} \to 25 = [det(\boldsymbol{A})]^2 \to det(\boldsymbol{A}) = \pm 5$

(c) $\boldsymbol{A}^{-1} = \frac{1}{det(\boldsymbol{A})}\boldsymbol{C}^T = \pm\frac{1}{5}\begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$

$\boldsymbol{A} = (\frac{1}{det(\boldsymbol{A})}\boldsymbol{C}^T)^{-1} = det(\boldsymbol{A}) \cdot (\boldsymbol{C}^T)^{-1}$

$= det(\boldsymbol{A}) \cdot \frac{1}{det(\boldsymbol{C^T})}\boldsymbol{C'}^T \quad$ ($\boldsymbol{C'}$ is cofactor matrix for $\boldsymbol{C^T}$)

$= \pm 5 \cdot \frac{1}{25}\boldsymbol{C'}^T = \pm\frac{1}{5} \cdot \begin{bmatrix} 10 & 15 & 5 \\ 5 & 10 & 10 \\ 10 & 10 & 15 \end{bmatrix}^T$

$\to \boldsymbol{A} = \pm \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(d) $det(2\boldsymbol{A}\boldsymbol{C}^T) = 2^3 \cdot det(det(\boldsymbol{A}) \cdot \boldsymbol{I}) = 8 \cdot [det(\boldsymbol{A})]^3 = \pm 1000$

4. (a) $det(\boldsymbol{A}) = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 17 & 9 \\ 64 & 289 & 81 \end{vmatrix} = (17-8)(9-8)(9-17) = -72$

$det(\boldsymbol{B_1}) = \begin{vmatrix} 1 & 1 & 1 \\ 7 & 17 & 9 \\ 49 & 289 & 81 \end{vmatrix} = (17-7)(9-7)(9-17) = -160$

$det(\boldsymbol{B_2}) = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 7 & 9 \\ 64 & 49 & 81 \end{vmatrix} = (7-8)(9-8)(9-7) = -2$

$det(\boldsymbol{B_3}) = \begin{vmatrix} 1 & 1 & 1 \\ 8 & 17 & 7 \\ 64 & 289 & 49 \end{vmatrix} = (17-8)(7-8)(7-17) = 90$

$x_1 = \frac{det(\boldsymbol{B_1})}{det(\boldsymbol{A})} = \frac{20}{9}$, $x_2 = \frac{det(\boldsymbol{B_2})}{det(\boldsymbol{A})} = \frac{1}{36}$, $x_3 = \frac{det(\boldsymbol{B_3})}{det(\boldsymbol{A})} = -\frac{5}{4}$

(b) $\frac{1}{2}\left|\begin{vmatrix} 8 & 64 & 1 \\ 17 & 289 & 1 \\ 9 & 81 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|\begin{vmatrix} 1 & 8 & 64 \\ 1 & 17 & 289 \\ 1 & 9 & 81 \end{vmatrix}\right| = \frac{1}{2}\left|\begin{vmatrix} 1 & 1 & 1 \\ 8 & 17 & 9 \\ 64 & 289 & 81 \end{vmatrix}\right| = \frac{1}{2}\left|-72\right| = 36$

5. (a) The eigenvalues of $A = 2$, 1 , -1

The eigenvectors of $A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

(b) $(A - cI)\bar{x} = (\lambda - cI)\bar{x}$

The eigenvalues of $A - cI = 2 - c, 1 - c, -1 - c$

The eigenvectors of $A - cI = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

(c) $P = [\bar{v}_1, \bar{v}_2, \bar{v}_3] \to B^{-1}AB = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$\xrightarrow{(\ \ )^T} = (B^{-1}AB)^T = D^T$

$\longrightarrow B^T A^T (B^T)^{-1} = D^T = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\xrightarrow{(B^T)^{-1}=C} C^{-1}A^T C = D$

$\because C = [\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3] = \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

Ans: The eigenvectors of $A^T = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

(d) $A^3 = BD^3B^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -5 & 3 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & 1 & -1 \\ 16 & 3 & -3 \\ 8 & 3 & -4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -5 & 3 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 16 & -3 & -2 \\ 24 & -1 & -6 \\ -3 & 9 & -7 \end{bmatrix}$$

6. For system A

(a) Let $\boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ Then $\frac{d}{dt}\mathbf{Y} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\mathbf{Y}$

(b) $\det\begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} = \mathbf{0} \quad (3-\lambda)^2 - 4 = 0 \quad \lambda = 5, 1$

$\lambda = 5, \quad \boldsymbol{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = 1, \quad \boldsymbol{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Let $\mathbf{Y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ and $\mathbf{Y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\mathbf{Y} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$

(c) $T = \lambda_1 + \lambda_2 = 5 + 1 = 6 \quad D = \lambda_1\lambda_2 = 5$

(d) Since $T > 0$ and $D > 0$, the system A is unstable.

For system B

(a) Let $\boldsymbol{u} = \begin{bmatrix} y' \\ y \end{bmatrix}$ $\boldsymbol{u'} = \begin{bmatrix} y'' \\ y' \end{bmatrix}$

Then $\boldsymbol{u'} = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$

(b) $\det \begin{bmatrix} -5-\lambda & -6 \\ 1 & -\lambda \end{bmatrix} = \mathbf{0}$ $(-\lambda)(-5-\lambda)+6=0$ $\lambda = -2, -3$

$\lambda = -2,\quad \boldsymbol{x_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\lambda = -3,\quad \boldsymbol{x_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Let $\boldsymbol{u} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-3t}$ and $\boldsymbol{u}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\boldsymbol{u} = \begin{bmatrix} y' \\ y \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{-2t} - 1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{-3t}$$

(c) $T = \lambda_1 + \lambda_2 = -2 - 3 = -5 \quad D = \lambda_1\lambda_2 = 6$

(d) Since $T < 0$ and $D > 0$, the system B is stable.

7. (a) Makov matrix: $\begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{2}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$,

eigenvalues:

$$\det \begin{bmatrix} \frac{1}{2}-\lambda & \frac{1}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{2}{5}-\lambda & \frac{3}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5}-\lambda \end{bmatrix} = \mathbf{0}$$

$$\rightarrow \frac{-1}{50}(\lambda-1)(5\lambda-1)(10\lambda-1) = 0$$

$$\rightarrow \lambda = 1,\ \frac{1}{5},\ \frac{1}{10}.$$

(b) For $\lambda = 1$, $\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{2}{5}-1 & \frac{3}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5}-1 \end{bmatrix} \mathbf{x} = 0 \rightarrow \mathbf{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

For $\lambda = \frac{1}{5}$, $\begin{bmatrix} \frac{1}{2}-\frac{1}{5} & \frac{1}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{2}{5}-\frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5}-\frac{1}{5} \end{bmatrix} \mathbf{x} = 0 \rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = \frac{1}{10}$, $\begin{bmatrix} \frac{1}{2}-\frac{1}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{2}{5}-\frac{1}{10} & \frac{3}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5}-\frac{1}{10} \end{bmatrix} \mathbf{x} = 0 \rightarrow \mathbf{x} = \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix}$.

The Markov matrix has three independent eigenvectors, thus it is diagonalizable.

(c) Solve $\begin{bmatrix} \frac{1}{2}-1 & \frac{1}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{2}{5}-1 & \frac{3}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5}-1 \end{bmatrix} \mathbf{x} = 0 \rightarrow \mathbf{x} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$.

(d) The absolute value of each element of $A^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$ should be less than 1% ($\frac{1}{100}$).

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ can be decomposed to $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix}$.

$$\begin{aligned}
\text{The error to the steady state} &= A^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \\
&= (1)^n \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} - \left(\frac{1}{5}\right)^n \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{1}{10}\right)^n \frac{2}{3} \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \\
&= -\left(\frac{1}{5}\right)^n \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{1}{10}\right)^n \frac{2}{3} \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix}
\end{aligned}$$

When n=3,

$$\begin{aligned}
&-\left(\frac{1}{5}\right)^3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{1}{10}\right)^3 \frac{2}{3} \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix} \\
=& \begin{bmatrix} \frac{23}{3000} \\ \frac{-1}{3000} \\ \frac{-11}{1500} \end{bmatrix}.
\end{aligned}$$

The error to the steady state is less than 1% after three years.