2017 Fall EE203001 Linear Algebra - Homework 2

Due: 2017/10/20

1. (8%) Use Gauss-Jordan elimination on $\begin{bmatrix} U & I \end{bmatrix}$ to find the upper triangular U^{-1}:

$$UU^{-1} = I \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

2. (12%) Suppose A is already lower triangular with 1 's on the diagonal. $A = L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$.

The elimination matrices E_{21} , E_{31} , E_{32} contain $-a$ then $-b$ then $-c$.
(a) Multiply E_{21} E_{31} E_{32} to find the single matrix E that produces $EA = I$.
(b) Multiply E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} to bring back L (nicer than E).

Solution:

$$E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$$

$$E = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

3. (10%) For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 5 & c & c \\ 3 & 4 & c \\ c & c & c \end{bmatrix}$$

Solution:

A is not invertible for
c=0(zero column)
c=5(equal rows)
c=4(equal columns)

4. (10%) A and B are symmetric across the diagonal (because 6=6,2=2). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 9 \end{bmatrix}$$

Solution:

$$A=\begin{bmatrix}2&6\\6&20\end{bmatrix}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}2&6\\0&2\end{bmatrix}\text{=LU=}\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}2&0\\0&2\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}\text{=LDU}$$

$$B=\begin{bmatrix}1&2&0\\2&5&2\\0&2&9\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}1&2&0\\0&1&2\\0&0&5\end{bmatrix}\text{=LU=}\begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&5\end{bmatrix}\begin{bmatrix}1&2&0\\0&1&2\\0&0&1\end{bmatrix}\text{=LDU}$$

$$UisL^T, LisU^T$$

5. (8%) Find A^{-1} and B^{-1}(if they exist) by elimination on $[A \quad I]$ and $[B \quad I]$:

$$A=\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$$

Solution:

For matrix A:

$$\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}^{-1}=\frac{1}{4}\begin{bmatrix}3&-1&-1\\-1&3&-1\\-1&-1&3\end{bmatrix}$$

For matrix B, observe that

$$\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

So B^{-1} does not exist.

6. (10%) We can actually solve $A\boldsymbol{x}=\boldsymbol{b}$ without needing explicit A. First, you are asked to solve $L\boldsymbol{c}=\boldsymbol{b}$ to find $\boldsymbol{c}$. Then solve $U\boldsymbol{x}=\boldsymbol{c}$ to find $\boldsymbol{x}$. Briefly explain the concept behind these steps.

$$L=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} \quad \text{and} \quad U=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} \quad \text{and} \quad \boldsymbol{b}=\begin{bmatrix}4\\5\\6\end{bmatrix}$$

Solution:

$$\boldsymbol{c}=\begin{bmatrix}4\\1\\1\end{bmatrix} \qquad \boldsymbol{x}=\begin{bmatrix}3\\0\\1\end{bmatrix}$$

Let the complete elimination matrix be E . Thus $EA=U, A=E^{-1}U=LU \Rightarrow L=E^{-1}$;
By solving $L\boldsymbol{c}=\boldsymbol{b} \Rightarrow \boldsymbol{c}=E\boldsymbol{b}$, we do the forward elimination part of solving $A\boldsymbol{x}=\boldsymbol{b}$.
By solving $U\boldsymbol{x}=\boldsymbol{c}=E\boldsymbol{b}$, we do the back substitution part of solving $A\boldsymbol{x}=\boldsymbol{b}$.

7. (12%) Given $Z=\begin{bmatrix}1&1\\0&-1\end{bmatrix}$. Compute (a)$Z-Z^T$ (b)Z^3 (c)Z^{2n} n=1,2,3,....

Solution:

$$(a) \quad Z-Z^T=\begin{bmatrix}1&1\\0&-1\end{bmatrix}-\begin{bmatrix}1&0\\1&-1\end{bmatrix}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

$(b) \quad Z^2 = ZZ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad , Z^3 = Z^2Z = (I)Z = Z$

$(c) \quad Z^{2n} = (Z^2)^n = I^n = I$

8. (8%) Let $Q = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$. Find Q^{-1}

Solution:

$$Q^{-1} = \frac{1}{cos\theta^2 + sin\theta^2} \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$$

9. (12%) Suppose you eliminate upwards (almost unheard of.) Use the last row to produce zeros in the column (the pivot is 1). Then use the second row to produce zero above the second pivot. Find the factors in the unusual order A =UL

$$\textbf{Upper times lower} \qquad \mathbf{A} = \begin{bmatrix} 6 & 3 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = L$$

then

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad , A = UL$$

10. (10%) Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? Multiplying A on the right by P_2 exchanges the ________ of A.

$$A = \begin{bmatrix} 0 & 0 & 9 \\ 4 & 8 & 7 \\ 0 & 4 & 1 \end{bmatrix}$$

Solution:

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

is upper triangular. Multiplying onthe right by a permutation matrix P_2 exchanges the columns. To make this A lower triangular,we also need P_1 to exchange rows2 and 3

$$P_1AP_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$