2017 Fall EE203001 Linear Algebra - Homework 1 Solution

1. (12%) Calculate the dot product $\vec{u} \cdot \vec{v}$ and $\vec{u} \cdot \vec{w}$ and $\vec{u} \cdot (\vec{v} + \vec{w})$ and $\vec{w} \cdot \vec{v}$.

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Solution:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= 1 \times 5 + 1 \times 6 \\ &= 11 \end{aligned} \quad \begin{aligned} \vec{u} \cdot \vec{w} &= 1 \times 2 + 1 \times 4 \\ &= 6 \end{aligned} \quad \begin{aligned} \vec{u} \cdot (\vec{v} + \vec{w}) &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \\ &= 17 \end{aligned} \quad \begin{aligned} \vec{w} \cdot \vec{v} &= 2 \times 5 + 4 \times 6 \\ &= 34 \end{aligned}$$

2. (14%) Normally 4 "plane" in 4-dimensional space meet at a _____. Normally 4 column vectors in 4-dimensional space can combine to produces b. What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $b = (1, 7, 4, 2)$? What 4 equations for x, y, z, t are you solving?

Solution:

Four planes in 4-dimensional space normally meet at a point. The solution to $A\mathbf{x}$ = (1,7,4,2) is $\mathbf{x}$ = (-6,3,2,2) if matrix A has columns (1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1), which means the columns of A produces b = (1,7,4,2) with coefficients -6,3,2,2.

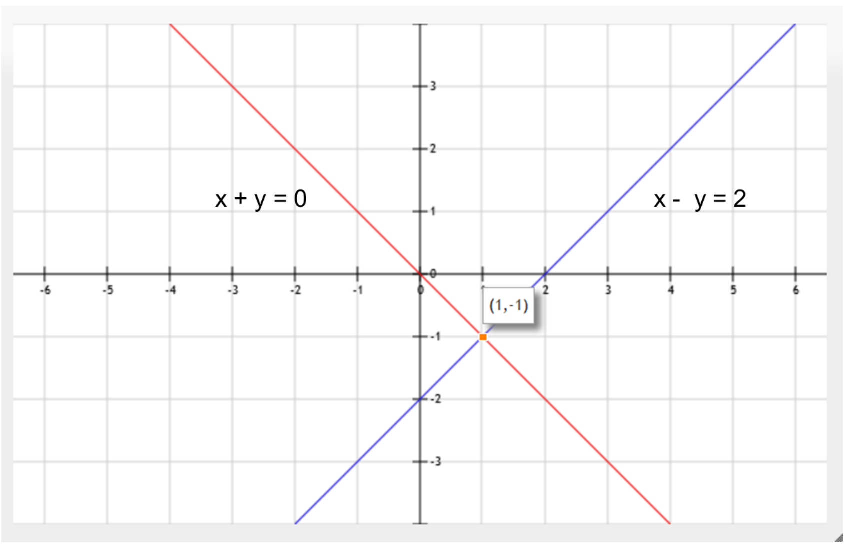
To obtain the coefficients, we should solve the following 4 equations:

$$\begin{cases} x + y + z + t = 1 \\ \quad\; y + z + t = 7 \\ \qquad\;\; z + t = 4 \\ \qquad\qquad t = 2 \end{cases}$$

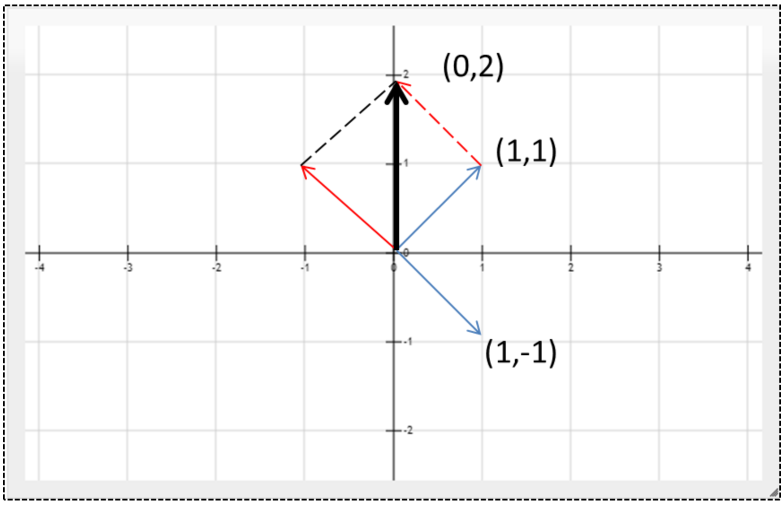
3. (12%) Draw the row and column pictures for the equations $x + y = 0$, $x - y = 2$.

Solution:

Row picture: solving for the intersection of the two equations



Column picture: $1 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \times \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$



4. (12%) Choose a coefficient b that makes this system singular. Then choose a right side g that makes it solvable. Find two solutions in that singular case.

$$x + by = 4$$
$$2x + 2y = g.$$

Solution:

The system becomes singular when b = 1 since $2x + 2y$ is 2 times $x + y$. Then $g = 8$ makes the lines become **the same**: there are infinitely many solutions like (4,0), (0,4).

5. (14%) In the xy plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y =$ _____ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equation if $c =$ _____.

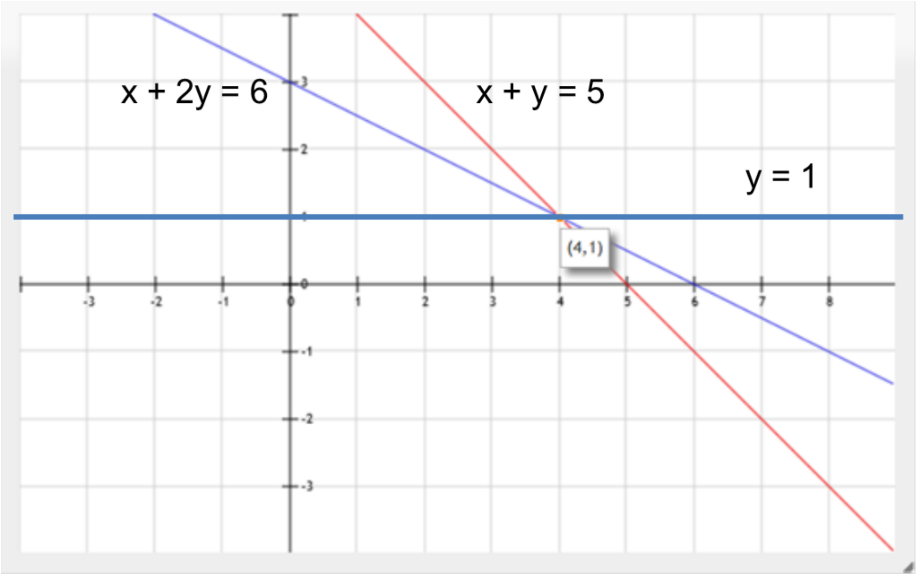
Solution:

We can do the elimination in matrix form:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Then we get the equation $y = 1$.



And substitude $x = 4, y = 1$ into $5x - 4y = c$, we have $c = 5 \times 4 - 4 \times 1 = 16$.

6. (12%) Which number q makes this system singular and which right side t gives if infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \\ 3y + qz &= t \end{aligned}$$

Solution:

Subtract the first equation from the second one, we have $3y - 4z = 5$; to make the system singular with infinitely many solutions, we must produce $0 = 0$ situation in the third equation, which implies $q = -4, t = 5$.
Then to find the solution that has $z = 1$, we substitude z with 1 and solve

$$\begin{cases} x + 4y = 3 \\ x + 7y = 12 \end{cases} \quad \Rightarrow \quad x = -9 \quad y = 3$$

Note that you may solve any two of the three equations to get the same result.

7. (12%) For which three numbers a will elimination fail to give three pivots?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \text{ is singular for three values of a.}$$

Solution:

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix} \longrightarrow \begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{bmatrix} \longrightarrow \begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{bmatrix}$$

Observe the resulting matrix we obtain three possibilities: a=0 (fail to give first pivot), a=2 (fail to give second pivot), a=4 (fail to give third pivot).

8. (12%) Apply elimination to the 3 by 4 augmented matrix $[Ab]$. How do you know this system has no solution? Change the last number 6 so there is a solution.

$$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}.$$

Solution:

Let the augmented matrix be A':

$$A' = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The lack of pivot and the contradiction of $0 = 3$ in the last row of the augmented matrix make the system singular with no solution. Changing the last number from 6 to 3 changes the situation to $0 = 0$ so that the system has infinitely many solutions.