2017 Fall EE203001 Linear Algebra - Homework 8 solution

Due: None

1. (0%)
   (a) What matrix transforms $(1,0)$ into $(2,5)$ and transforms $(0,1)$ to $(1,3)$?
   (b) What matrix transforms $(2,5)$ to $(1,0)$ and $(1,3)$ to $(0,1)$?
   (c) Why does no matrix transform $(2,6)$ to $(1,0)$ and $(1,3)$ to $(0,1)$?

   Solution:

   (a) $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = inverseof\,(a)$

   (c) $A\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ must be $2A\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

2. (0%)
   (a) What matrix M transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u)?
   (b) What matrix N transforms (a,c) and (b,d) to $(1,0)$ and $(0,1)$?
   (c) What condition on a,b,c,d will make part (b) impossible?

   Solution:

   (a) $M = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u); this is the "easy" direction.

   (b) $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ transforms in the inverse direction, back to the standard basis vectors.

   (c) $ad = bc$ will make the forward matrix singular and the inverse impossible.

3. (0%) Find the singular value decomposition of A = $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

   Solution:

   $A = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ has rank 1 and$A^TA = [5]$ and $\sigma_1 = \sqrt{5}$ is the only sigular value in $\Sigma = \begin{bmatrix} \sqrt{5} \\ 0 \\ 0 \end{bmatrix}$

   $AA^T = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has $\lambda = 5,0,0$ and $u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

4. (0%) Continue above question, Please find the pseudoinverse of A and the least square solution $\hat{x}_0$ that $b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for $A\hat{x}_0 = b$

   Solution:

   $$A^+ = V\Sigma^+U^T = [5]\begin{bmatrix} \sqrt{5} & 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4\sqrt{5} & 2\sqrt{5} & 0 \end{bmatrix}$$

   $A\hat{x}_0 = b => \hat{x}_0 = A^+b => 4\sqrt{5}$

5. (0%) The parabola $\mathbf{w_1} = \frac{1}{2}(x^2 + x)$ equals one at $x = 1$, and zero at $x = 0$ and $x = -1$. Find the parabolas $\mathbf{w_2}, \mathbf{w_3}$ from the conditions given below and then find $y(x)$ using $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ by linearity.

   (a) $\mathbf{w_2}$ equals one at $x = -1$, and zero at $x = 0$ and $x = 1$.

   (b) $\mathbf{w_3}$ equals one at $x = 0$, and zero at $x = 1$ and $x = -1$.

   (c) $y(x)$ equals 9 at $x = 1$ and 5 at $x = 0$ and 7 at $x = -1$.

   Solution:

   Through simple calculation of just by observation, we have $\mathbf{w_2} = \frac{1}{2}(x^2 - x)$ and $\mathbf{w_3} = 1 - x^2$, and by linearity we have $y(x) = 9\mathbf{w_1} + 5\mathbf{w_3} + 7\mathbf{w_2}$.

6. (0%) Suppose T is reflection across the 45° line, and S is reflection across the y axis. If $\mathbf{v} = (a, b)$, find $S(T(\mathbf{v}))$ and $T(S(\mathbf{v}))$.

   Solution:

   T takes (a, b) to (b, a) and S takes (a, b) to $(-a, b)$, thus $S(T(\mathbf{v})) = (-b, a)$ and $T(S(\mathbf{v})) = (b, -a)$. This shows that generally $ST \neq TS$.

7. (0%) Suppose a linear T transforms (1,1) to (1,0,1) and (2,3) to (1,-1,4). Find $T(\mathbf{v})$:

   (a) $v = (3, 4)$

   (b) $v = (4, 6)$

   (c) $v = (a, b)$

   Solution:

$$T(\mathbf{x}) = A\mathbf{x} = A\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 4 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}$$

   (a) $T(\mathbf{v}) = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

   (b) $T(\mathbf{v}) = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 8 \end{bmatrix}$

   (c) $T(\mathbf{v}) = \begin{bmatrix} 2 & -1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ a - b \\ -a + 2b \end{bmatrix}$

8. (0%) Given $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$.
   If $T(M) = AMB$, please find $T^{-1}(M)$ in the form (   )M(   ).

   Solution:

   Because A and B are invertible, $T(T^{-1}(M)) = M$
   So $T^{-1}(M) = A^{-1}MB^{-1}$

$$T^{-1}(M) = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} & M & \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$