2017 Fall EE203001 Linear Algebra - Homework 7 solution

Due: 2017/12/22

1. (10%) Compute A^TA and AA^T and their eigenvalues and unit eigenvectors for V and U

$$\textbf{Rectangular} \quad \textbf{matrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution:

$AA^T = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ has $\sigma_1^2 = 7$ with $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\sigma_2^2 = 3$ with $u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$A^TA = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ has $\sigma_1^2 = 7$ with $v_1 = \begin{bmatrix} \frac{2}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}$ and $\sigma_2^2 = 3$ with $v_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ and $v_3 = \begin{bmatrix} \frac{1}{\sqrt{21}} \\ -\frac{2}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{bmatrix}$

Then $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$

2. (10%) Suppose $(T(v_1) = w_1 + 2w_2 + 3w_3$ and$T(v_2) = 2w_2 + 3w_3$ and$T(v_3) = 3w_3$. Find the matrix A for T using these basis vectors. What input vector $v gives T(v) = w_1$

Solution:

The matrix A for T is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix}$

For output $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ choose $v = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} => \begin{bmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = v$

3. (10%) Show that A and B are similar by finding M so that $B = M^{-1}AM$:

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ 6 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & -2 \\ 4 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 9 & 4 \\ -2 & 3 \end{bmatrix}$

Solution:

$A = S_A \Lambda_A S_A^{-1}$

$B = S_B \Lambda_B S_B^{-1}$

If A is similar to B, then $\Lambda_A = \Lambda_B$

$\Lambda_A = S_A^{-1} A S_A$, $B = S_B \Lambda_A S_B^{-1} = S_B(S_A^{-1} A S_A) S_B^{-1} = M^{-1}AM$

(a)

$A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$$

$$M = S_A S_B^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$M = S_A S_B^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

4. (10%) Find the eigenvalues and unit eigenvectors v_1, v_2 of $A^T A$. Then find $u_1 = Av_1/\sigma$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } A^T A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \text{ and } AA^T = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

Verify that u_1 is a unit eigenvectors of AA^T. Complete the matrices U, Σ, V.

Solution:

$det(A^T A - \lambda I) = 0, \lambda = 10, 0$

$$\lambda = 10, u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \lambda = 0, u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$det(AA^T - \lambda I) = 0, \lambda = 10, 0$

$$\lambda = 10, v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \quad \lambda = 0, v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$$

Then $A = U\Sigma V^T$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$$

We can find that $u_1 = Av_1/\sigma$.

5. (10%)

   (a) If U and V are unitary matrices, show that U^{-1} and UV are also unitary.

   (b) A is a matrix with independent columns. Show that $A^H A$ is not only Hermitian but also positive definite.

   Solution:

   (a) $U^H U = I, U-1(U^H)^{-1} = U-1(U^{-1})^H = I \Rightarrow U^{-1}$ is unitary. Also, $(UV)^H(UV) = V^H U^H UV = I \Rightarrow UV$ is unitary.

   (b) $(A^H A)^H = A^H A^{HH} = A^H A$. By the definition of definite positive, we check $(\mathbf{z}^H A^H)(A\mathbf{z}) = \|A\mathbf{z}\|^2$, which is positive unless $A\mathbf{z} = 0$. Since A has indpendent columns, $A\mathbf{z} = 0$ only if $\mathbf{z} = 0 \Rightarrow A^H A$ is positive definite.

6. (10%) If A is a Hermitian matrix,show the property of its' real and imaginary part.(symmetric, Hermitian, ...etc.) **Solution:**

   Let $A = R + iS = (R + iS)^H = R^T - iS^T \Rightarrow$ the real part is symmetric while the imaginary part is **skew-symmetric**.

7. (10%) Which classes of matrices does P belong to: invertible, Hermitian, unitary? Compute P^2, P^3, and P^{100}. What are the eigenvalues of P?

$$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix}.$$

   Solution:

   This P is invertible and unitary. $P^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $P^3 = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix} = -iI$.
   Then $P^{100} = (-i)^{33} P = -iP$. The eigenvalues of P are the roots of $\lambda^3 = -i$, which are i and $i \exp^{2\pi i/3}$ and $i \exp^{4\pi i/3}$.

8. (10%) Compute $\mathbf{y} = F_8\mathbf{c}$ by the three FFT steps for $\mathbf{c} = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation for $c = (0, 1, 0, 1, 0, 1, 0, 1)$.

   Solution:

   $\mathbf{c} \to (1, 1, 1, 1, 0, 0, 0, 0) \to (4, 0, 0, 0, 0, 0, 0, 0) \to (4, 0, 0, 0, 4, 0, 0, 0) = F_8\mathbf{c}$.
   $\mathbf{c} \to (0, 0, 0, 0, 1, 1, 1, 1) \to (0, 0, 0, 0, 4, 0, 0, 0) \to (4, 0, 0, 0, -4, 0, 0, 0) = F_8\mathbf{c}$.

9. (10%) Prove that if $\boldsymbol{A}$ is a real symmetric matrix, then all eigenvalues of $\boldsymbol{A}$ are real numbers.

   Solution:

$$\begin{aligned} & \boldsymbol{Ax} &= \boldsymbol{\lambda}\, \boldsymbol{x} \\ \to & (\boldsymbol{Ax})^{\boldsymbol{H}} &= (\boldsymbol{\lambda}\, \boldsymbol{x})^{\boldsymbol{H}} \\ \to & \boldsymbol{x}^{\boldsymbol{H}} \boldsymbol{A}^{\boldsymbol{H}} &= \overline{\boldsymbol{\lambda}}\, \boldsymbol{x}^{\boldsymbol{H}} \\ \to & \boldsymbol{x}^{\boldsymbol{H}} \boldsymbol{A} &= \overline{\boldsymbol{\lambda}}\, \boldsymbol{x}^{\boldsymbol{H}} \end{aligned}$$

$\rightarrow\ \boldsymbol{x^H A^H x = \overline{\lambda}\, x^H x}$

$\rightarrow\ \boldsymbol{\lambda\, x^H x = \overline{\lambda}\, x^H x}$

$\rightarrow\ \boldsymbol{(\lambda - \overline{\lambda})\|x\|^2} = 0$

$\rightarrow\ \because \boldsymbol{x \neq 0} \quad \therefore \boldsymbol{\|x\|^2 \neq 0}$

$\rightarrow\ \boldsymbol{\lambda = \overline{\lambda}} \rightarrow \boldsymbol{\lambda}$ is real

10. (10%) The columns of the Fourier matrix F are the *eigenvectors* of the cyclic permutation P. Multiply PF to find the eigenvalues λ_1 to λ_2:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}$$

This is $PF = F\Lambda$ or $P = F\Lambda F^{-1}$.The eigenvector matrix(usually S) is F.

Solution:

$det(P - \lambda I) = \lambda^4 - 1$
$\rightarrow \lambda = 1, i, i^2 = -1, i^3 = -i$